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Editorial Article

Reply to ”Various issues around the L_1-norm distance”

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Abstract - A distance function between two random variables or vectors was proposed in 2003 in a Ph.D. dissertation [1, 2]. Initially called a *probability metric*, it is now known as ”Łukaszyk-Karmowski metric” or LK-metric and has been successfully applied in various fields of science and technology. It does not satisfy the identity of indiscernibles (*Leibniz's law*) axiom of the metric, the ontological axiom also invalidated by the ugly duckling theorem. This note addresses two false claims made in a preprint [3] that LK-metric is the same as the mean absolute difference and that it is ill-defined. The fallacy of the first claim is straightforward: the mean absolute difference is defined solely for independent and identically distributed random variables, contrary to LK-metric. Thus, if one considers $E|X-X|$, then the random variable X must be *independent of itself*, which implies its degenerate probability distribution and $E|X-X|=0$. If X has a degenerate probability distribution, then Y, which is identically distributed as X, also has a degenerate probability distribution and $E|X-X|=0=E|X-Y|$, invalidating the second claim.

Keywords - Łukaszyk-Karmowski metric; mean absolute difference; identity of indiscernibles

The following two false claims concerning a distance function of two random variables or vectors, known as ”Łukaszyk-Karmowski metric” or LK-metric [1, 2] were raised in a preprint [3] published solely on arXiv:

1. LK-metric is actually the functional
$$\tilde{\psi}([X]_d,[Y]_d)=E|X-Y|, \tag{1}$$
where $E|X-Y|$ is the mean absolute difference between random variables X and Y, which renders this functional ill-defined, and consequently that

2. when the author of LK-metric ”asserts that the identity of indiscernibles property is not realized by the metric $E|\cdot-\cdot|$ he uses that we end up understanding he implicitly identifies the identically distributed random variables of $\mathcal{L}_1(\mathbb{R})$. In other words, he reasons as if (1) were well-defined.”

These claims are based on the following two false arguments.

Argument 1. The ”so-called Lukaszyk-Karmowski metric used in mechanical physics or in quantum physics” is the expression ”of the statistical distance $E|X-Y|$ between two random variables X and Y”, the *mean absolute difference* between X and Y, also known as the *absolute mean difference* and the *Gini mean difference*.

Reply 1. This is not true. LK-metric $D_{*\star}(X,Y)$ between two random variables or vectors X and Y, where ”$*\star$” in the subscript stands for the types of probability distributions (normal, uniform, binomial, Laplace [4], etc.) of X and Y, is not the same as the mean absolute difference:
$$D_{*\star}(X,Y)\neq E|X-Y|. \tag{2}$$

$E|X-Y|$ is defined for independent and identically distributed (i.i.d.) random variables. Recall that two random variables X and Y assuming values in $I\subseteq\mathbb{R}$ having the cumulative distribution functions (cdf) $F_X(x)=\Pr(X\leq x)$ and $F_Y(y)=\Pr(Y\leq y)$ and joint cdf $F_{X,Y}(x,y)=\Pr(X\leq x\wedge Y\leq y)$ are identically distributed iff
$$F_X(x)=F_Y(x)\quad\forall x\in I, \tag{3}$$

and independent iff

$$F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y) \quad \forall x, y \in I, \tag{4}$$

which implies that they have the same type of probability distribution, and their means (μ) and standard deviations (σ) are the same ($\mu_x = \mu_y$ and $\sigma_x = \sigma_y$).
On the other hand, random variables or random vectors X and Y, being the arguments of LK-metric can have different types of probability distributions, different means, and/or different standard deviations (violating the definition (3)) and can be dependent (violating the definition (4)).
In most practical applications of LK-metric (see e.g., [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56]), usually X and Y were considered independent, had the same type of probability distributions, and the same standard deviations. However, their means were considered different ($\mu_x \neq \mu_y$). It was shown, for example, that in modeling of intratumor heterogeneity the consequences of random switching between (stable) coexisting point attractors of different relative depth can be readily quantified using LK-metric which specifies geometric distance of the points with coordinates μ_x and μ_y known up to the respective probability distributions [18]. This approach is physically based, allowing the real uncertainty in the location of the sample point around its location to be considered [6, 7].
Furthermore, if one considers $E|X - X|$, then by the definition (4) the random variable X must be *independent of itself* and the only way a random variable X can be *independent of itself* is if for every measurable set A either its probability $\Pr(X \in A) = 1$ or $\Pr(X \in A) = 0$ [57]. In other, words, the definition (4) requires the indempotence of $\Pr(X \in A)$

$$\Pr(X \in A) \cdot \Pr(X \in A) = \Pr(X \in A) \Leftrightarrow \Pr(X \in A) \in \{0, 1\}, \tag{5}$$

which in turn implies that the random variable X has a one-point distribution in the discrete case or a Dirac delta distribution in the continuous case. In both cases, X describes almost surely (a.s.) equal event. In other words, X has a degenerate distribution and $E|X - X| = 0$.
In the case X has a degenerate distribution δ, LK-metric vanishes for $\mu_x = \mu_y$, the same as $E|X - X|$

$$E|X - X| = D_{\delta\delta}(X, X)_{\mu_x = \mu_y} = 0. \tag{6}$$

However, for any other non-degenerate probability distribution $*$ of X

$$D_{**}(X, X)_{\mu_x = \mu_y} > 0, \tag{7}$$

and is well-defined, since in this case X is not *self-independent*, and thus degenerate random variable, which is implied by the definition (4) leading to the property (5), but represents the same physical phenomenon, which is independently observed [58] by two distinct observers (X, X). A rainbow is a perfect illustration of observer dependence [59].
This characteristic non-zero distance effect *built in* LK-metric allows to avoid ill-conditioning problems in radial basis function interpolation [60, 29] and inverse distance weighting [61, 31, 36, 48], where the interpolation accuracy can be improved by choosing the type of distance metric [27, 48]. Incorporating *random mechanics* into prediction [62] leads to a smooth interpolation function [19]. These spatial interpolation methods do not require statistical assumptions [63].
By preventing zero distances based on parameter uncertainty [4], LK-metric can, furthermore, be used in analysis of nondeterministic dynamical systems with competing attractors [53]. Since LK-metric represents the mean of distances between all the outcomes of the two uncertain *objects*, it can also be used in uncertain nearest neighbor classification [64, 65]. The actual value of an uncertain *object* is modeled by a probability density function [66].
By the property (7), LK-metric does not satisfy the identity of indiscernibles (*Leibniz's law*) axiom of the metric [9, 11, 67, 68] stating that there cannot be separate *objects* that have all their properties in common (no two distinct *objects* are equally similar): "X and X" in (7) have all their properties (type of the distribution, μ_x, and σ_x) in common, but can also be independently observed [58]. LK-metric is not the only distance function that does not satisfy the identity of indiscernibles axiom. The partial metric [69] axioms, for example, also allow that each *object* not necessarily have to have zero distance from itself. However, the partial metric satisfies two additional axioms of *small self-distances* and *modified triangle inequality*, which are not satisfied by LK-metric

[70]. Further theoretical considerations concerning LK-metric can be found in a considerable literature in this field (see e.g. [71, 72, 73, 74, 33, 75, 76, 68, 77, 67, 78, 70, 79]).
Remarkably, the identity of indiscernibles ontological *axiom*, introduced to philosophy by Gottfried Wilhelm Leibniz around 1686, was invalidated by the ugly duckling theorem stated in 1969 [80, 81] and asserting that every two *objects* one perceives are equally similar (or equally dissimilar).

Argument 2. Some ”authors using $E|X-Y|$ have fallen into the trap of identifying within $\mathcal{L}_1(\mathbb{R})$ the identically distributed random variables rather than the almost surely equal random variables. Unfortunately, this leads to a logical impasse. Seeking a contradiction, suppose that we set (1), where $X, Y \in \mathcal{L}_1(\mathbb{R})$ are identically distributed without being almost surely equal. We have $[X]_d = [Y]_d$ and $E|X-Y| > 0$ (since $E|X-Y| = 0 \Leftrightarrow X \overset{a.s.}{=} Y$), and we end up with the following contradiction:

$$\begin{aligned} \tilde{\psi}([X]_d, [X]_d) &\overset{(1)}{=} E|X-X| = 0 < \\ &< E|X-Y| \overset{(1)}{=} \tilde{\psi}([X]_d, [Y]_d) = \tilde{\psi}([X]_d, [X]_d), \end{aligned} \tag{8}$$

meaning that (1) is ill-defined”.

Reply 2. This is not true. Let us extract the main argument of the inequality (8)

$$E|X-X| = 0 \overset{?}{<} E|X-Y|. \tag{9}$$

Indeed $E|X-X| = 0$, as we have shown in Reply 1, since here X must be *self-independent* random variable, which is implied by the definition (4). But since X and Y are i.i.d. they also need to satisfy both definitions (3) and (4). Then, if X is *self-independent* random variable with degenerate distribution, then also Y is and therefore $X \overset{a.s.}{=} Y$. Thus, also $E|X-Y| = 0$. Therefore, the inequality (8) turns into the equality

$$\begin{aligned} \tilde{\psi}([X]_d, [X]_d) &\overset{(1)}{=} E|X-X| = 0 = \\ &= E|X-Y| \overset{(1)}{=} \tilde{\psi}([X]_d, [Y]_d) = \tilde{\psi}([X]_d, [X]_d), \end{aligned} \tag{10}$$

and the contradiction or *logical impasse* that it allegedly brought collapses.
Now assume, as does the author of [3], that X and Y are i.i.d. and $X \overset{a.s.}{\neq} Y$. Then the last equality in (8) is false, $\tilde{\psi}([X]_d, [Y]_d) \neq \tilde{\psi}([X]_d, [X]_d)$. If in the relation (8)

1. $\tilde{\psi}([X]_d, [X]_d) \overset{(1)}{=} E|X-X| = 0$ (LHS of (8)), and

2. $\tilde{\psi}([X]_d, [Y]_d) \overset{(1)}{=} E|X-Y| > 0$ (RHS of (8)), since, as we have shown above and as the author of [3] correctly assumes, $E|X-Y| = 0 \Leftrightarrow X \overset{a.s.}{=} Y$, then consequently

3. $\tilde{\psi}([X]_d, [Y]_d) > \tilde{\psi}([X]_d, [X]_d)$.

Therefore, even under the false assumption of X and Y being i.i.d. and not a.s. equal, which is required to form the inequality (8), the contradiction that this inequality allegedly introduces also collapses.

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