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Editorial Article

Metallic Ratios and Angles of a Real Argument

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Abstract - We extend the concept of metallic ratios to the real argument $n \in \mathbb{R}$ considered as a dimension by analytic continuation showing that they are defined by an argument of a normalized complex number, and for rational $n \neq \{0, \pm 2\}$, they are defined by Pythagorean triples. We further extend the concept of metallic ratios to metallic angles.

Keywords - - Metallic ratios; Metallic angles; Pythagorean triples; Emergent dimensionality; Mathematical physics.

1 Introduction

Each rectangle contains at least one square with an edge h equal to the shorter edge of the rectangle. If a rectangle contains n such squares and its edges $nh + d$ and h satisfy

$$M(n) := \frac{nh + d}{h} = \frac{h}{d} \tag{1}$$

they satisfy a metallic ratio; the golden ratio for $n = 1$, the silver ratio for $n = 2$, shown in Fig.1, the bronze ratio for $n = 3$, etc.

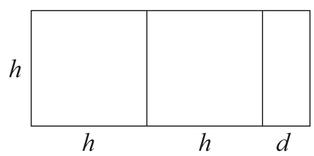

Figure 1: Silver rectangle and ratio $M(2)_+ = (2h + d)/h = h/d$.

Solving the relation (1) for $M(n)$ leads to the quadratic equation

$$M(n)_\pm^2 - nM(n)_\pm - 1 = 0, \tag{2}$$

having roots

$$M(n)_\pm = \frac{n \pm \sqrt{n^2 + 4}}{2}, \tag{3}$$

shown in Fig. 2 for $n \in \mathbb{R}$.

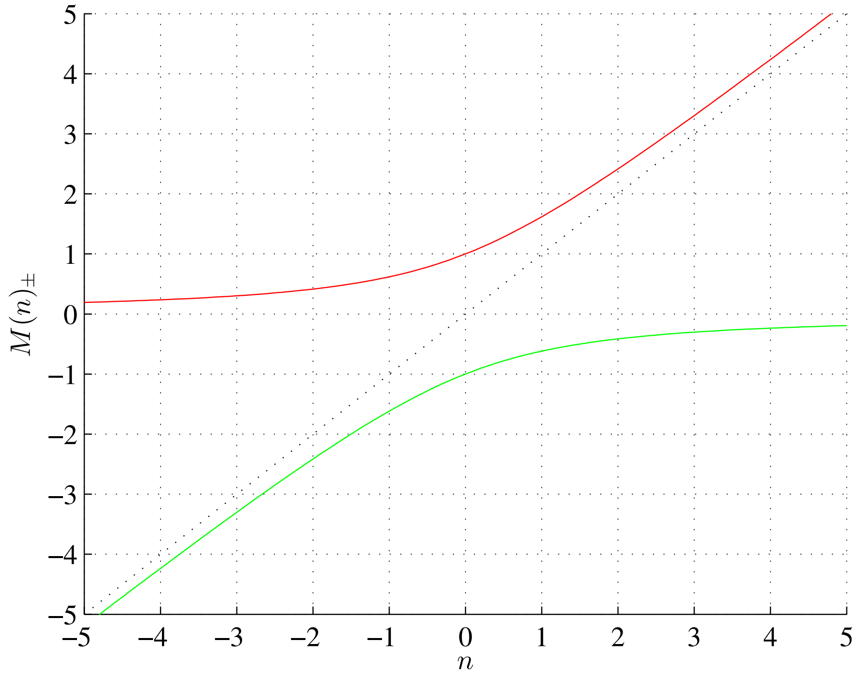

Figure 2: Metallic ratios: positive $M(n)_+$ (red), negative $M(n)_-$ (green) as continuous functions of $n \in \mathbb{R}$ for $-5 \leq n \leq 5$.

Metallic ratios (3) have interesting properties, such as

$$\begin{aligned} &1. \quad M(n)_- M(n)_+ = -1, \\ &2. \quad M(n)_- + M(n)_+ = n, \\ &3. \quad -M(-n)_- = M(n)_+, \quad \text{or} \\ &4. \quad M(n)_\pm = \pm e^{\operatorname{arcsinh}(\pm n/2)}. \end{aligned} \tag{4}$$

Furthermore, as $n \to \infty$, the factor $+4$ in the square root becomes negligible, and $M(n)_\pm \approx \{n, 0\}$ for large n.

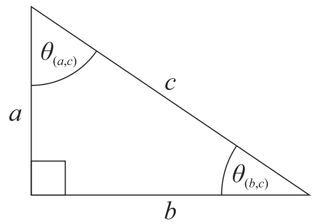

Figure 3: Right triangle showing a longer (b), shorter (a) hypotenuse, catheti (c) and angles $\theta = \theta_{(b,c)}$ and $\theta_{(a,c)}$.

It was shown [1] that for $n \neq \{0, 2\}$ positive metallic ratios (3) can be expressed by primitive Pythagorean triples, as

$$M(n)_+ = \cot\left(\frac{\theta}{4}\right), \tag{5}$$

and for $n \geq 3$

$$n = 2\sqrt{\frac{c+b}{c-b}}, \tag{6}$$

where θ is the angle between a longer cathetus b and hypotenuse c of a right triangle defined by a Pythagorean triple, as shown in Fig.3, whereas for $n = \{3, 4\}$ it is the angle between a hypotenuse and a shorter cathetus a ($\{M(3)_+, M(10)_+\}$ and $\{M(4)_+, M(6)_+\}$ are defined by the same Pythagorean triples, respectively, $(5, 12, 13)$ and $(3, 4, 5)$), and

$$M(1)_+ = \cot\left(\frac{\pi - \theta_{(3,5)}}{4}\right). \tag{7}$$

For example, the Pythagorean triple $(20, 21, 29)$ defines $M(5)_+$, the Pythagorean triple $(3, 4, 5)$ defines $M(6)_+$, the Pythagorean triple $(28, 45, 53)$ defines $M(7)_+$, and so on.

Since the edge lengths of a metallic rectangle are assumed to be nonnegative, generally only the positive principal square root $M(n)_+$ of (2) is considered. However, the nonnegativity of distances (corresponding to the ontological principle of identity of indiscernibles) does not hold for the LK-metric [2], for example; such an axiomatization is misleading [3]. Furthermore, fractal dimensions have been verified to be consistent with experimental observations [4,5] which justifies the analytic continuation of metallic ratios to the real argument n considered as a dimension [6,7]. This is discussed in Section 2. Section 3 extends the concept of metallic ratios to metallic angles of a real argument n. Section 4 concludes the findings of this study.

2 Metallic Ratios of a Real Number

The metallic ratio $M(n)_\pm$ of $n \in \mathbb{R}$ is defined by an acute angle of a right triangle $0 < \theta < \pi/2$.

Proof. We express the RHS of the equation (5) using half-angle formulas and substituting $\varphi := \theta/2$

$$\cot\left(\frac{\theta}{4}\right) = \cot\left(\frac{\varphi}{2}\right) = \frac{1+\cos\varphi}{\sin\varphi} = \frac{1+\cos\left(\frac{\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} = \\ = \frac{1+\operatorname{sgn}\left(\cos\left(\frac{\theta}{2}\right)\right)\sqrt{\frac{1+\cos\theta}{2}}}{\operatorname{sgn}\left(\sin\left(\frac{\theta}{2}\right)\right)\sqrt{\frac{1-\cos\theta}{2}}} = M(n)_+, \tag{8}$$

since $0 < \theta < \pi/2$ (we exclude degenerated triangles), so $\operatorname{sgn}(\sin(\theta/2)) = \operatorname{sgn}(\cos(\theta/2)) = 1$. Multiplying the numerator and denominator of (8) by $\sqrt{(1+\cos\theta)/2}$ and performing some basic algebraic manipulations, we arrive at the quadratic equation for $M(\theta)$

$$\sin(\theta)\, M(\theta)^2 - 2\,[1+\cos(\theta)]\, M(\theta) - \sin(\theta) = 0, \tag{9}$$

having roots

$$M(\theta_+)_\pm = \frac{(1+\cos(\theta_+)) \pm \sqrt{2\,(1+\cos(\theta_+))}}{\sin(\theta_+)}, \tag{10}$$

corresponding to the metallic ratios (3) for $0 < \theta_+ < \pi/2$. □

We can extend the domain of Theorem 2 by analytic continuation to $0 < \theta_+ < \pi$ as $\operatorname{sgn}(\sin(\theta_+/2)) = \operatorname{sgn}(\cos(\theta_+/2)) = 1$ in this range. However, extending it further to $-\pi < \theta_- < 0$ we note that in this range $\operatorname{sgn}(\sin(\theta_-/2)) = -1$. Thus, the quadratic equation (9) becomes

$$\sin(\theta_-)\, M(\theta_-)^2 + 2\,[1+\cos(\theta_-)]\, M(\theta_-) - \sin(\theta_-) = 0, \tag{11}$$

and its roots are

$$M(\theta_-)_\pm = \frac{-(1+\cos(\theta_-)) \pm \sqrt{2\,(1+\cos(\theta_-))}}{\sin(\theta_-)}. \tag{12}$$

The metallic ratio of $n \in \mathbb{R}$ is defined by an angle $-\pi < \theta \leq \pi$.

Proof. Equating relations (3) and (10) and solving for n gives

$$n_+ = \frac{2\,(1+\cos(\theta_+))}{\sin(\theta_+)}, \tag{13}$$

for $0 < \theta_+ < \pi$. This identity can also be obtained directly from the second property (4) applied to the ratio (10). Solving the relation (13) for θ_+ yields

$$\frac{n_+ + 2i}{n_+ - 2i} = e^{i\theta_+} = \cos\theta_+ + i\sin\theta_+ = \frac{a}{c} + \frac{b}{c}i := z(n_+), \tag{14}$$

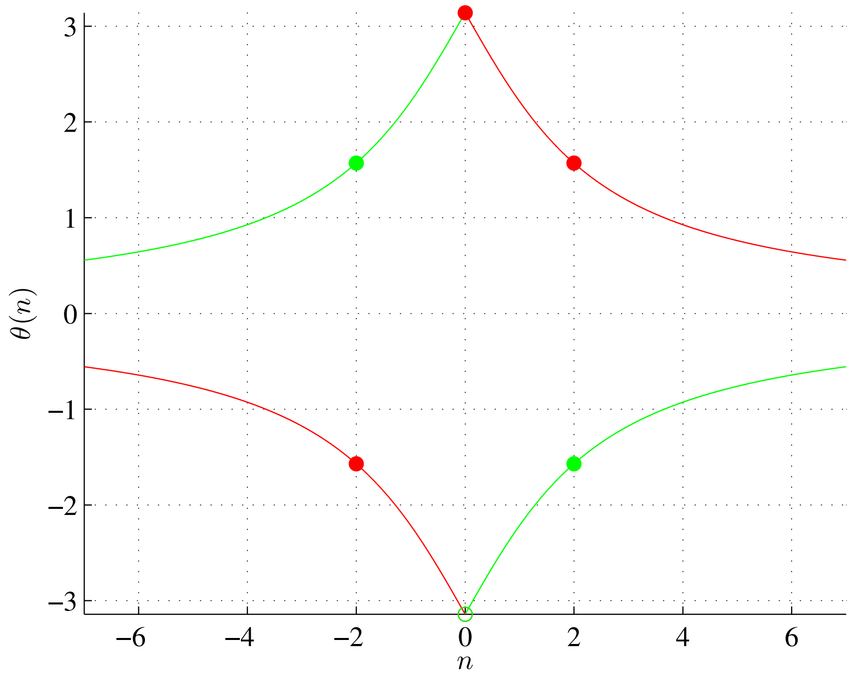

Figure 4: Phases of the complex number $z(n)$ (red, n_+) and its conjugate $\overline{z(n)}$ (green, n_-) for $-7 \le n \le 7$. $\theta(\pm 2) = \pm\pi/2$, $\theta(0) = \pi$.

Similarly, applying the second property (4) to the ratio (13) gives

$$n_- = \frac{-2\,(1+\cos(\theta_-))}{\sin(\theta_-)}, \tag{15}$$

for $-\pi < \theta_- < 0$. Solving the relation (15) for θ_- yields

$$\frac{n_- - 2i}{n_- + 2i} = e^{-i\theta_-} = \frac{a}{c} - \frac{b}{c}i := z(n_-) = \overline{z(n_+)}, \tag{16}$$

as a conjugate of the relation (14). The relations (14) and (16) remove the singularity of $\theta = l\pi, l \in \mathbb{Z}$ in the relations (13), (15), and $\lim_{n_+ \to \pm\infty} \arg(z(n_+)) = \lim_{n_+ \to \pm\infty} \arg(\overline{z(n_+)}) = 0$. □

Equations (14) and (16) relate $n_\pm \in \mathbb{R}$ which defines a metallic ratio (3) to the normalized complex number $z(n_+)$. The angles $\theta_+ = \arg(z(n_+))$ and $\theta_- = \arg(\overline{z(n_+)})$ are shown in Fig. 4. There are two axes of symmetry.

In summary, metallic ratios as functions of θ are

$$M(\theta_\pm)_\pm = \frac{\pm\,(1+\cos(\theta_\pm)) \pm \sqrt{2\,(1+\cos(\theta_\pm))}}{\sin(\theta_\pm)}, \tag{17}$$

where the first "±" defines the range of θ and the second "±" corresponds to the positive or negative form of the ratio. Therefore, the first and second properties (4) hold for $M(\theta_\pm)_- M(\theta_\pm)_+ = -1$ and $M(\theta_\pm)_- + M(\theta_\pm)_+ = n_\pm$ but the third property (4) holds as $-M(-\theta_\pm)_\pm = M(\theta_\pm)_\pm$.

Fig. 5 shows metallic ratios (10) and (12) as functions of $-1.2\pi \le \theta \le 1.2\pi$, $\theta_+ = \arg(z(n_+))$, and $\theta_- = \arg(\overline{z(n_+)})$.

For $n \ne \{0, \pm 2\}, n \in \mathbb{Q}$, the triple $\{a, b, c\}$ corresponding to the angle θ (14), (16) is a Pythagorean triple.

Proof. Plugging rational $n :- l/m, m \ne 0, l, m \in \mathbb{Z}$ into the relation (14) gives

$$\frac{l^2 - 4m^2}{l^2 + 4m^2} + \frac{4lm}{l^2 + 4m^2}i = \frac{a}{c} + \frac{b}{c}i, \tag{18}$$

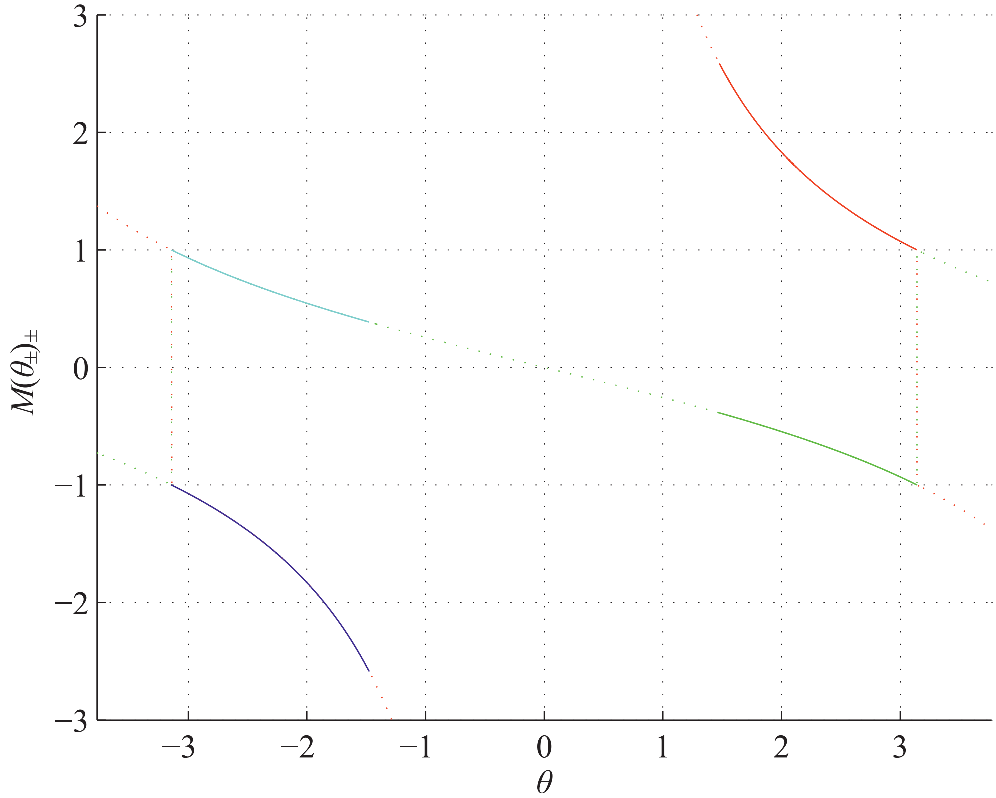

Figure 5: Metallic ratios: positive $M(\theta_+)_+$ (red), negative $M(\theta_+)_-$ (green) as a function of $-1.2\pi \le \theta \le 1.2\pi$ (dotted), and $\theta = \arg(z(n_+))$ (solid), and $\theta = \arg(\overline{z(n_+)})$ (positive: solid blue; negative: solid cyan) for $0 \le n_+ \le 2.2$.

and $a = l^2 - 4m^2, b = 4lm, c = l^2 + 4m^2, a, b, c \in \mathbb{Z}$ is a possible solution. It is easy to see that $a^2 + b^2 = c^2$, which is valid $\forall l, m \neq 0 \in \{\mathbb{R}, \mathbb{I}\}$. $n = 0$ implies $l = 0$ and $a = -c = -4m^2, b = 0$; $n = \pm 2$ implies $l = \pm 2m$ and $a = 0, b = \pm 8m^2, c = 8m^2$. □

Table 1 shows the generalized Pythagorean triples that define the metallic ratios for $n = \{0.1, 0.2, \dots, 7\}$. For $n = \{-7, -6.9, \dots, -0.1\}$ set $b \leftrightarrow -b$.E.g. for $n = -7$, $\{45, 28, 53\} \leftrightarrow \{45, -28, 53\}$.

For $\hat{n} :- n(n+2)/(n+1), n \in \mathbb{R}$ the positive metallic ratio $M(\hat{n})_+ = n + 1$.

Proof. Direct calculation of the defining relation (3) for $\hat{n}$. Furthermore, $n = \left(\hat{n} - 2 + \sqrt{\hat{n}^2 + 4}\right)/2$. □

For example, for $n = 1$, $\hat{n} = 3/2$, and $M(\hat{n})_+ = 2$. The numerator sequence $n(n+2)$ is the OEIS A005563 entry. For such $\hat{n}$, Theorem 2 provides:

$$\begin{aligned}
\hat{k} &:= n + 1, \\
z &:= (\hat{k} + i)^4, \\
a &= \mathrm{Re}(z) = \hat{k}^4 - 6\hat{k}^2 + 1 \quad \text{(OEIS A272870)}, \\
b &= \mathrm{Im}(z) = 4(\hat{k}^3 - \hat{k}) \quad \text{(OEIS A272871)}, \\
c^2 &:= a^2 + b^2, \\
c &= \pm(\hat{k}^4 + 2\hat{k}^2 + 1) \quad c_+ = \text{OEIS A082044},
\end{aligned} \tag{19}$$

shown in Fig. 6. a and c are even and b is odd function of $\hat{k}$ defined by the relation (19). We note that $n = -1$, where $a = \pm 1, b = 0$, and $c = \pm 1$ is a dimension of the void, the empty set $\emptyset$, or (-1)-simplex.

3 Metallic Angles of a Real Number

We can extend the concept of metallic ratios (1) to angles as

$$\frac{n(2\pi - \varphi) + \varphi}{2\pi - \varphi} = \frac{2\pi - \varphi}{\varphi}, \tag{20}$$

Table 1: Pythagorean triples associated with metallic ratios for rational $n = \{0.1, 0.2, \ldots, 7\}$.

n	a	b	c	n	a	b	c
0.1	-399	40	401	3.6	28	45	53
0.2	-99	20	101	3.7	969	1480	1769
0.3	-391	120	409	3.8	261	380	461
0.4	-12	5	13	3.9	1121	1560	1921
0.5	-15	8	17	4	3	4	5
0.6	-91	60	109	4.1	1281	1640	2081
0.7	-351	280	449	4.2	341	420	541
0.8	-21	20	29	4.3	1449	1720	2249
0.9	-319	360	481	4.4	48	55	73
1	-3	4	5	4.5	65	72	97
1.1	-279	440	521	4.6	429	460	629
1.2	-8	15	17	4.7	1809	1880	2609
1.3	-231	520	569	4.8	119	120	169
1.4	-51	140	149	4.9	2001	1960	2801
1.5	-7	24	25	5	21	20	29
1.6	-9	40	41	5.1	2201	2040	3001
1.7	-111	680	689	5.2	72	65	97
1.8	-19	180	181	5.3	2409	2120	3209
1.9	-39	760	761	5.4	629	540	829
2				5.5	105	88	137
2.1	41	840	841	5.6	171	140	221
2.2	21	220	221	5.7	2849	2280	3649
2.3	129	920	929	5.8	741	580	941
2.4	11	60	61	5.9	3081	2360	3881
2.5	9	40	41	6	4	3	5
2.6	69	260	269	6.1	3321	2440	4121
2.7	329	1080	1129	6.2	861	620	1061
2.8	12	35	37	6.3	3569	2520	4369
2.9	441	1160	1241	6.4	231	160	281
3	5	12	13	6.5	153	104	185
3.1	561	1240	1361	6.6	989	660	1189
3.2	39	80	89	6.7	4089	2680	4889
3.3	689	1320	1489	6.8	132	85	157
3.4	189	340	389	6.9	4361	2760	5161
3.5	33	56	65	7	45	28	53

where for $n = 1$ well known golden angle $\varphi(1)_- \approx 2.4$, shown in Fig. 7, is obtained.

Solving the relation (20) for φ leads to the quadratic equation

$$n\varphi(n)_\pm^2 - 2\pi(n+2)\varphi(n)_\pm + 4\pi^2 = 0, \tag{21}$$

having roots

$$\frac{\varphi(n)_\pm}{\pi} = \frac{n + 2 \pm \sqrt{n^2+4}}{n}, \tag{22}$$

shown in Fig. 8.

In this case, both their products and sums

$$\begin{aligned} \frac{\varphi(n)_-\varphi(n)_+}{\pi^2} &= \frac{4}{n}, \\ \frac{\varphi(n)_- + \varphi(n)_+}{\pi} &= 2\frac{n+2}{n}, \end{aligned} \tag{23}$$

are dependent on n, where $\lim_{n\to\pm\infty} \varphi(n)_-\varphi(n)_+ = 0$ and $\lim_{n\to\pm\infty} \varphi(n)_- + \varphi(n)_+ = 2\pi$.

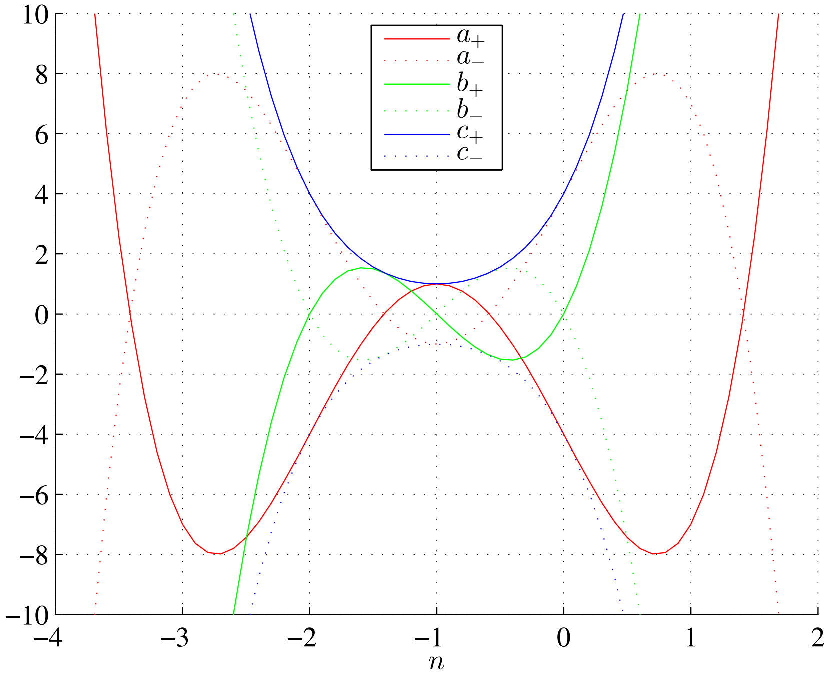

Figure 6: The triples $\{a, b, c\}$ corresponding to the angle θ (14), (16) as functions of $n = \left(\hat{n} - 2 \pm \sqrt{\hat{n}^2 + 4}\right)/2$. The positive metallic ratio $M(n(n+2)/(n+1))_+ = n + 1$ for $n \in \mathbb{R}_+$.

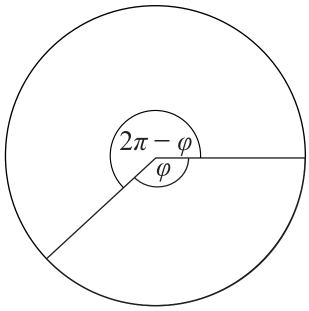

Figure 7: Golden angle $\varphi(1)_- = \pi(3 - \sqrt{5})$.

The positive metallic ratios (3) are equal to the positive metallic angles (22) for

$$\hat{n}_1 = \frac{4\pi(\pi + 1)}{2\pi + 1} \approx 7.1459, \tag{24}$$

where

$$M(\hat{n}_1)_+ = \varphi(\hat{n}_1)_+ = 2\pi + 1 \approx 7.2832, \tag{25}$$

the negative metallic ratios (3) are equal to the positive metallic angles (22) for

$$\hat{n}_2 = \frac{-1 + \sqrt{8\pi + 1}}{4\pi} - \frac{\sqrt{8\pi + 1}}{2} - \frac{1}{2} \approx -2.7288, \tag{26}$$

where

$$M(\hat{n}_2)_- = \varphi(\hat{n}_2)_+ = \frac{-\sqrt{8\pi + 1} - 1}{2} \approx -3.0560, \tag{27}$$

and the positive metallic ratios (3) are equal to the negative metallic angles (22) for

$$\hat{n}_3 = \frac{-1 - \sqrt{8\pi + 1}}{4\pi} + \frac{\sqrt{8\pi + 1}}{2} - \frac{1}{2} \approx 1.5696, \tag{28}$$

where

$$M(\hat{n}_3)_+ = \varphi(\hat{n}_3)_- = \frac{\sqrt{8\pi + 1} - 1}{2} \approx 2.0560. \tag{29}$$

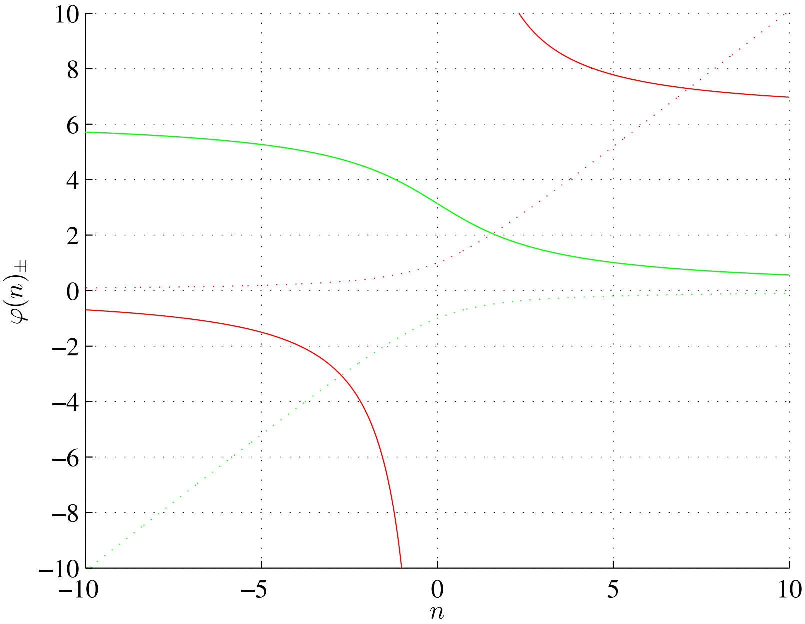

Figure 8: Metallic angles (solid) and ratios (dotted), positive (red), negative (green), as continuous functions of $-10 \leq n \leq 10$.

4 Conclusions

The positive golden ratio (3) and the negative golden angle (22) are observed in nature. In flower petals, sunflowers and pinecones, tree branches, shells' shapes, spiral galaxies, hurricanes, reproductive dynamics, etc. But why has nature chosen $n = 1$ corresponding to the complex number $z(1) = (-3 + 4i)/5$ (14) remains to be researched. We note that $\{3, 4, 5\}$ forms the smallest Pythagorean triple, which hints at the relation of such a nature's choice to the second law of thermodynamics.

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