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Invited Article

On the intimate association between even binary palindromic words and the Collatz-Hailstone iterations

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Abstract - The celebrated $3x + 1$ problem is reformulated via the use of an analytic expression of the trailing zeros sequence resulting in a single branch formula $f(x) + 1$ with a unique fixed point. The resultant formula $f(x)$ is also found to coincide with that of the discrete derivative of the sorted sequence of fixed points of the reflection operator on even binary palindromes of fixed even length $2k$ in any interval $[0 \cdots 2^{2k} - 1]$. A set of equivalent reformulations of the problem are also presented.

Keywords - Collatz conjecture; Hailstone sequences; Binary palindromes; Iterative dynamics.

1 Introduction

The so called, Hailstone sequences are the iterates of a discrete branched dynamical system over positive integers that were first introduced by Lothar Collatz [?] at 1937. It became alternatively known as the *′3x+1′* problem due to the particular formula used in one of its branches.

Lately, the notion of a generic *′3x+1′* semigroup was also introduced [?][?]. At 1972, Conway had already generalized the problem in the generic form $(a_i x + b_i), 0(mod b_i)$ in a manner that allowed him to create a programming language called *′fractran′* which was of the same power with a universal Turing machine thus being able to derive standard undecidability results[?].

A very recent review of the problem has been given by Lagarias[?]. Also, recent results on connections of this problem with cellular automata appeared which are reviewed in [?] together with previous similar attempts. Mostly, strong associations with Wang tiling machines have been introduced in the work of Sterin[?] with possible connections to biological complexity.

An analysis of the resulting Hailstone sequences of iterants from a physicist's perspective appeared in [?] and [?]. It is hoped that the preset analysis will also be of interest for other physically inspired toy models for stochastic and fractal processes.

In the next section, an appropriate reformulation of the Hailstone iteration is introduced which makes use of a special function also associated with the so called *dyadic valuation*[?]. This allows transforming the original branched process to a single branch one with a unique fixed point.

In section 3, a set of necessary definitions are introduced for palindromic words or palindromes based on fixed maximal length binary expansions as fixed points of the reflection group over such expansions. A hierarchical construct is used to extract certain scaling maps associating each expansion length L with its next one across different intervals of exponential length revealing the underlying tree structure of such patterns.

Certain properties of these hierarchies of patterns are discussed and a crucial property is proven that provides a direct link with the original Hailstone process.

In section 4, the role of palindromes and the associated reflection group is discussed revealing an interesting type of interaction between a form of 'mirror' images inside the main process.

Furthermore, two indices in the form of binary probability measures are proposed for the study of the conjectured global convergence, associated with both the inner reflective structure as well as the internal complexity of the binary patterns produced by the Hailstone process.

2 Reformulation of the Collatz-Hailstone (CH) iteration

The standard, or $3x+1$ Collatz-Hailstone process is defined via the branched map

$$x_{n+1} = \begin{cases} x_n/2, & 0(mod2) \\ 3x_n + 1, & 1(mod2) \end{cases} \tag{1}$$

Let us introduce the two auxiliary maps $f_0(x) = x/2$ and $f_1(x) = 3x + 1$. It is obvious that any invocation of f_0 would cause this iteration to enter a cycle any time x_n reaches a power of two since it would then remain on the first branch until it reaches 1 with the second branch mapping $1 \to 4$ immediately after.

On the other hand, f_1 always maps odd integers to even integers thus any iteration stays at the second branch only once while looping over the first branch until all powers of two divisors are exhausted.

The particular sequence of all integers after removal of its 2 factors is already known as the odd part sequence, catalogued as *A000265* in the OEIS database[**?**]

The resulting symbolic dynamics of branch execution then is intimately related with binary divisibility or equivalently, the amount of zeros present at the start of any binary expansion. This is another well known sequence in computer science under the name of *trailing zeros*(TZS), also catalogued as *A007814* [**?**]. Any integer is then represented as $x = \sigma_{odd}(x)2^{t(x)}$ where $t(x)$ denotes specific values of the TZS.

In terms of the run length analysis of symbolic sequences[**?**],[**?**] where every bit string of length L is represented by an alternating polynomial, the TZS corresponds to the zero order coefficient for all compressed binary expansions. This will be presented in more detail in the next sections.

When represented as a sequence over all integers, the TZS is equivalent to the so called, *2-adic valuation*, the first of the *π-adic* valuations corresponding to the expression of all the exponents of prime factorization as sequences[**?**]. The structure of TZS encodes a special tree graph which in the context of word combinatorics is related to the *Zimin words* or, more generally *sequipowers*[**?**]. It is then abstractly similar to the celebrated fractal *ABACABA* sequence[**?**].

The simplest approach to obtain a concrete formula for computing the TZS utilises the binary divisibility by successive powers of two so that one can write

$t(x) = \sum_{i=0}^{l_2(x)} 1_\chi(mod(x, 2^i) = 0)$

where $l_2(x) = 1 + \lfloor log_2(x) \rfloor$ is the binary logarithm standing for the maximal power of two of the expansion of x. Similar expression will also arise in higher alphabets in the more general setting posed by Conway's fractran. In the case of the binary alphabet, it is also possible to rewrite the same using the Hamming distance between x and $x-1$ as prescribed in the relevant OEIS page[**?**]. A graphical representation of the $2^{t(x)}, x \in [0, ...,]$ is shown in figure 1.

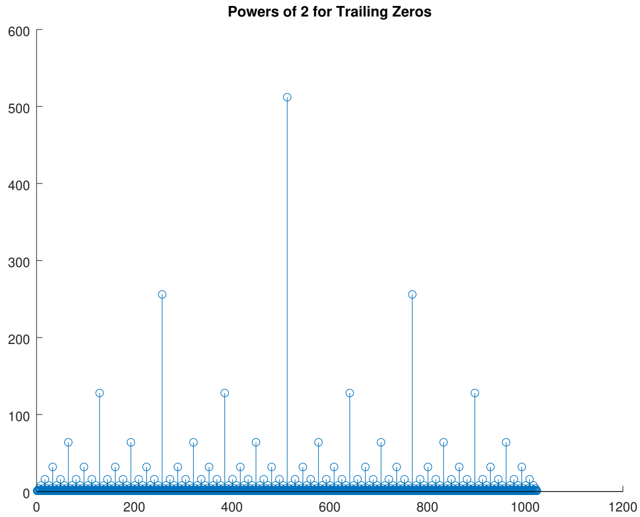

Figure 1: The tree structure of the TZS associated even factors

Since, knowledge of a complete shift in the exponents of every 2 factor is possible beforehand, it should also be possible to rephrase the original problem so that any computation would spend only a single step to each of the two branches given a complete knowledge of the total shift $2^{-t(x)}$ thus effectively realising the whole bunch of $f_0^{t(x)}$ total application of the first map.

Such a transcription is facilitated by rewriting a composite which takes into account both cases of $f_0^n \circ f_1$ and $f_1 \circ f_0^m$ at once by noticing the equivalence of f_1 with $3x + mod(x, 2)$ in which case the original CH in (1) is rewritten as:

$$x_{n+1} = 2^{-t(x_n)}(3x + mod(x_n, 2)) + 1 - mod(x_n, 2) \tag{2}$$

The expression in (2) automatically sends to either of the two branches depending on the $(mod2)$ class. It is preferable to rewrite it also in the form

$$x_{n+1} = A(x_n)x_n + B(x_n)$$

where $A(x_n) = 3/2^{t(x_n)}$ and $B(x_n) = 1 + mod(x_n, 2)(2^{-t(x_n)} - 1)$.

It is immediately obvious that $B(x_n) = 1$ for all cases. This is simply because of the complementarity of $t(x)$ with $mod(x, 2)$ since all roots of $t(x)$ are odd integers. Therefore the final reduction of CH in (1) is equivalent to the expression

$$x_{n+1} = \left(\frac{3}{2^{t(x_n)}}\right) x_n + 1 \tag{3}$$

This final form will exactly reproduce the same elements of the original iterations that do not contain successive binary shifts. A possible termination condition for this type of iteration can be given as $mod(log2(x_n, 2), 1) = 0$.

The coefficient that appears in (3) is of special importance and stands for the bridge between the original problem and that of the palindromic binary strings as explained in the next section. The fixed points of the final map in (3) are found via the standard condition $f(x) - x = 0$ rewritten as

$$\left(1 - \frac{1}{x}\right) 2^{t(x)} = 3 \tag{4}$$

Given the structure of the TZS for any expansion of length L in a maximal L interval $[0, ..., 2^L - 1]$ it holds that $0 \leq t(x) \leq L-1$. Restricting search in all powers of 2, $(x = 2^l : t(x) = l)$ immediately cancels out the exponential term in (3) leaving only the condition $2^l - 1 = 3$ so that the only possible integer root in (4) is at $x = 4$.

In the next section some appropriate definitions are introduced which will make it possible to establish the connections of the new single branch map with the issue of palindromic words in fixed length binary expansions.

3 Hierarchies of Palindromes

3.1 Preliminary definitions

The particular construct presented requires the introduction of constant length binary expansions for all words inside an interval. When expressed this way, all binary patterns inside an exponential interval are said to form a so called, *'Hamming Space'* the reasoning being that all such expansion are then forming a norm, linear vector space the norm being given be the Hamming distance[**?**].

All such representations require that any binary expansions are also characterized by a number of leading zeros. This leads to an additional ambiguity with the definition of certain operators acting on words like reflections or mirror inversions and the construction of palindromic words due to the need for additional parametrization for the expansion length required.

Because of this necessary to setup a different than usual representation which is only possible across a self-similar hierarchy of lexicographically ordered sets that can be represented as special asymmetric matrices of all patterns. To do this, the following terminology will be useful.

A number $M(L) = 2^L - 1$ is to be called a *Mersenne* number and an interval $s(L) = [0, ..., M(L)]$ is to be called a *Mersenne interval*. A self-similar sequence of intervals

$$s(1) \subset s(2) \subset \cdots \subset s(L) \subset \cdots$$

is to be associated with a set of $L \times 2^L$ matrices of lexicographically ordered bit patterns as a representation of each S_L in $1-1$ correspondence with the binary expansion of the row index $j \in s(L)$ via the polynomial representation.

The particular choice is justified by a variety of reasons including the fact that the above is also a well formed hierarchy of closures for certain binary operators like the *bit-wise XOR* which is known to have the group property.

It is also a known fact that each column of any S_L matrix representation is identical with the paths of a symmetric, homogeneous rooted binary tree thus corresponding to a self-similar hierarchy of binary tree structures.

Equivalently, the same can be phrased as an arithmetic equivalent of a hierarchy of *Hamming Cubes* or, subspaces of an L-dimensional hypercube, due to the fact that every element of a Hamming space associated with a fixed length binary expansion can be put into a one-to-one association with the edges of such a hypercube[**?**]

The particular form of the hierarchy of lex-ordered matrices is also known in another context as a set of *Orthogonal Designs*[**?**] when written in the equivalent $\{\pm 1\}$ alphabet instead of $\{0, 1\}$.

To further facilitate an exchange between the language of sequences and binary patterns of constant length, it is appropriate to denote $|w| \in \mathbb{N}$ for the arithmetic value of each binary word via the use of an "encoding" map

$$p : |w| = p(w) = \sum_{i=0}^{L-1} a_i 2^i$$

and its abstract "decoding" inverse

$$p^{-1} : [a_0, ..., a_L] \overset{p^{-1}}{\leftarrow} |w|$$

This helps establishing a direct association of the hierarchy of intervals with the hierarchy of matrices as

$$\begin{array}{ccccc} s(0) & \xrightarrow{\delta} & s(1) & \xrightarrow{\delta} & s(2)\cdots \\ & & \Big\downarrow{\scriptstyle p^{-1}} & & \Big\downarrow{\scriptstyle p^{-1}} \\ [0] & \longrightarrow & S_1 & \xrightarrow{\Delta} & S_2\cdots \end{array}$$

The additive action of the δ map is to simply increase the cardinality of any previous ordered list representing any Mersenne interval by applying to each member the same rule and perform a list concatenation like

$$\delta s(L) \to s(L+1) = [s(L), s(L) + 2^L], L = 0, 1, \cdots$$

Correspondingly, each new application of the decoder P^{-1} results in an equivalent self-similar action denoted by Δ comprising a concatenation of a copy the previous $S(L)$ matrix followed by the addition of a new top row of precisely L zeros to be followed by a left to right flip of its not-complement like:

$$S_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \to S_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Thus, the whole hierarchy comprises a sequence of internal 2-complements and reflections.

Effectively, all rows of any member matrix of the hierarchy is a periodic sequence spanning an exponential sequence of periods that can be directly computed via either a Boolean or an equivalent arithmetic formula as:

$$S_{i,j\in[0,\cdots,2^L-1]}(L) = 2^{-i}(j \otimes 2^i) = mod\left(\lfloor \frac{j}{2^i} \rfloor, 2\right), i = 0, 1, \cdots, L-1 \quad (5)$$

where $\otimes$ stands for the bit-wise AND operation.

Moreover, one may consider the action of arbitrary automata $\mathcal{A}$ accepting some or all of the rows of each member S_L. There are two main classes of possible automata described as either a)*Indicators* or maps $\mathcal{A} : S_L \to \{0, 1\}$ asserting existence of some property of a binary pattern or b)*Transducers*: $\mathcal{A} : S_L \to S_N$.

Restricting attention to such automata or their equivalent Turing machine expressions that halt, one may introduce an upper bound as $max(L, N)$ such that every such halting automaton represents an endomorphism in S_{Lmax}. It is then possible to project the action of all such automata into a new sequence or production via:

$$s_{\mathcal{A}} \leftarrow \left(p \circ \mathcal{A} \circ p^{-1}\right) s(L) \quad (6)$$

This implies the reduction of any such computation into a chain of maps producing a hierarchy of sequences of exponentially increasing length

$$\begin{array}{ccccccc}
s(0) & \xrightarrow{\delta} & s(1) & \xrightarrow{\delta} & s(2) & \xrightarrow{\delta} & \cdots \\
 & & \downarrow{\scriptstyle p^{-1}} & & \downarrow{\scriptstyle p^{-1}} & & \\
[0] & \longrightarrow & \mathcal{A}[S_1] & \xrightarrow{\Delta} & \mathcal{A}[S_2] & \xrightarrow{\Delta} & \cdots \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\
s_{\mathcal{A}}(0) & \xrightarrow{g} & s_{\mathcal{A}}(s(1)) & \xrightarrow{g} & s_{\mathcal{A}}(s(2)) & \xrightarrow{g} & \cdots
\end{array}$$

Computability of the action of the new g map in terms of simple arithmetic formulas is a separate and difficult issue in general for arbitrary automata. Whenever possible, one may exchange the action of any such automaton with the resulting sequence via a recursive list concatenation or even the generating function of the resulting sequence if sum-able at all.

A trivial example can be given in case of an automaton of which the action equals some permutation π of symbols across $w \in S_L$ in which case one can apply a 'transfer' principle based on the constant expansion length as:

$$\sum_{i=0}^{L-1} a_{\pi(i)} 2^i \cong \sum_{i=0}^{L-1} a_i 2^{\pi(i)}$$

For instance, inversion of all positions resulting in a mirror inversion of all words will also result in a map of the original sequence of natural numbers in an iterative sequence given by:

$$r(i+1) \leftarrow [r(i), r(i) + 2^{L-i-1}], r(0) = 0$$

Another simpler alternative of scaling maps for reflections is presented in the next section.

Notably, the sequence of applications of any such scaling map follows a well known pattern of another fundamental fractal sequence the so called, *Sum-of-Digits* sequence[**?**] as:

$$I\, g\, g\, g^{(2)}\, g\, g^{(2)}\, g^{(2)}\, g^{(3)}, \cdots, g^{sd(n)}$$

.

following the pattern:

$$0\,1\,1\,2\,1\,2\,2\,3\,\cdots$$

The $sd(n)$ sequence also admits the simplest arithmetic scaling map $g(x) = x + 1$ following the linear staircase of the natural lengths or maximal powers of two present in any lexicographically ordered set of binary patterns.

Sequences for which the leading zeros may play a role in their definition will not admit as simple a recursion as above and it will often exhibit a similar recursive structure with a branched map acting differently on the first and second part of a list concatenation scheme in the abstract form.

$$g(s(i+1)) \leftarrow [(g_0)(s(i), i), (g_1)(s(i), i)]$$

where the iteration index i may have to be explicitly included in general. Equivalently, the resulting compositions can always be extracted by a symbolic binomial expansion $(g_0 + g_1)^L$. The particular case of fixed length reflections and palindromes is analyzed in the next section.

3.2 Palindromes and the fixed length reflection group

Let w a binary word and $\mathcal{R}(w, L) : S_L \rightarrow S_L$ also denoted as R_L heretofore, an order reversing map also called the *reflector* heretofore, with S_L the set of all 2^L binary strings of same length L.

Then if $w = [a_0, \cdots, a_L]$, its mirror inversion or reflection is denoted as $\mathcal{R}_L(w) = [a_L, \cdots, a_0]$. $\mathcal{R}_L$ is then one of the two natural involutions for any words in every S_L the other being the 2-complement.

The hierarchical construction of the previous section imposes a discrimination between even and odd order palindromes depending on L being even or odd as well. Thus for all odd order matrices $S(2k+1)$ a palindrome may leave the "central" symbol at $k+1$ unaltered and only invert the order of the first k symbols so that:

$$a_{2k+1} = a_0, ..., a_{2k+1-i} = a_i, ..., a_{k+2} = a_k, i = 0, 1, ..., k$$

This class will not be treated further heretofore.

While the 2-complement is fixed point free in any S_L, the reflection operation will admit a set of fixed points known as palindromic words or simply, palindromes.

By construction the number of fixed points of every even order S_{2k} must necessarily contain the whole reflected set in S_k since anyone of them can get reflected thus forming a member of the set of fixed points of the reflector operator over any S_{2k}. Hence, the cardinality of the sets of fixed points must also form a sequence of 2^k fixed points of the $\mathcal{R}_{2k}$ action over any S_{2k} across the hierarchy. Obviously the 'edges' of each set comprising the all zeros and all ones patterns are always fixed points.

From now on the notation $\mathcal{R}_L(|w|)$ will be interpreted as the expanded form of (6), that is $\left(p \circ \mathcal{R} \circ p^{-1}\right)(|w|)$ which acts on the valuation of the word w by expanding, processing and again contracting to a new integer. Thus the total action over the sequence of natural numbers inside any Mersenne interval will always result in a new sequence in that same interval parameterized by the additional length parameter L.

By definition, a reflector has the group property since it sends one to one, any integer inside the same Mersenne interval thus being equivalent to a permutation. This is one of the main reasons for using the hierarchy over fixed length expansions. Otherwise, any power of two would be mapped to a one after bit order reflection thus failing to be a bijection.

The reflector does not act homomorphically over the standard arithmetic addition and multiplication but it does obey an anti-homomorphism with respect to the arithmetic equivalent of concatenation

$$\mathcal{R}_L(|x| + |y|2^L) = \mathcal{R}_L(|y|) + \mathcal{R}_L(|x|)2^L$$

The difference of fixed length reflections with leading zeros becomes more evident by noticing the appearance of a bit shift in any newly formed sub-sequence of reflected integers due to inversion of position of the leading zeros blocks.

Thus, a fixed length reflection can also be subsumed via the use of two scaling maps corresponding to the g map of the chain diagram of previous section leading to two different iterative list concatenation methods as

$$r(s(L+1)) \leftarrow [g_0 r(s(L)), g_1 r(s(L))], r(s(0)) = 0$$

where now $g_i(x) = 2x + mod(i, 2)$ applied point-wise across all previous list elements.

Before coming in the subject of fixed length palindromes and their properties it is important to add another toolbox in the description of fixed length binary expansion property of every fixed length reflected binary expansion which binds them with the TZS as well as the leading zeros sequence or LZS in a particularly useful way.

For all such binary words, an alternative compressed representation exists given in terms of a *run-length* encoding in the form of an alternating polynomial given as a bijective map

$rl(w) : S_L \leftrightarrow \mathbb{Z}^L : [a_0, \cdots, a_{L-1}] \leftrightarrow [\pm c_0(|w|) \cdots, \pm c_m(|w|)]$

under the convention of a minus sign for a block of zeros and vice versa. Each coefficient c_i counts the length of a block of same symbols marking with a $\mp$ sign whether it is a zero or one respectively.

An additional constraint over all alternating coefficients results from the fixed length expansions in the form

$$\sum_{i=0}^{m(n)} |c_i| = L, n \in s(L)$$

Due to the bijective nature of this mapping all distinct *integer partitions*[**?**] of L exist inside the set of RL representations of the members of any S_L since any such can always be turned to a binary pattern of same length.

The first and last of these coefficients, when expressed as sequences over all integers in the associated Mersenne interval are of special importance. In particular, the TZS is equivalent to all negative values of the first coefficient at even indices $c_1(2k)$, and zero for all even indices while the leading zeros sequence (LZS) is identified with $c_m(n), n \in s(L)$.

The latter is naturally associated with the binary logarithm $l_2(n)$ via $c_m(n) = L - l_2(x)$ in accord with the fixed maximal length representation used here.

The three fundamental sequences characterizing each binary pattern given by the triplet l_2, t_n, ds_2 share the same range in $[0, \cdots, L]$. Moreover, the TZS and the max. bit sequence l_2 share the same multiplicities of values.

The following proposition can also be proven:

Let t_n be a sequence of all $t(n), n \in s(L)$ and let r_n the sequence of reflected binary expansions as integers over the same interval. Then for all L, the second is a decreasing order sorting permutation of the first or $(t_z \circ r)(n) \cong sort_>(t_z(n))$.

This is a trivial result of the simultaneous fixed length reflection of all binary words in any interval which affects an exchange of the first and last coefficients in the RL representation and hence of the TZS with the LZS.

The corresponding permutation then is characterized by the first blocks of zeros being already sorted in size due to the fact that any leading zeros will have a difference from the maximal power of two that scales as $L - i$ with every new exponential sub-interval $s(i)$ which adds a single bit on top of all the previous expansions.

As a result, the permuted integers will contain a decreasing number of all even integers with 2^{L-i-1} factors when lexicographically ordered on the first half of the interval with all odd integers also mapped to the second half carrying over all zero values of the TZS.

Subsequent composition with the standard form of the TZS is then equivalent to a sorted counting of the number of unique digits in a TZS sequence of the form

1 0 2 0 1 0 3 0 1 0 2 0 1 0 $\cdots$

The result of such a composition of sequences is shown in figure 2 on all integers in $[0, \cdots 2^{10}]$ in a semilog graph to make the staircase structure more pronounced.

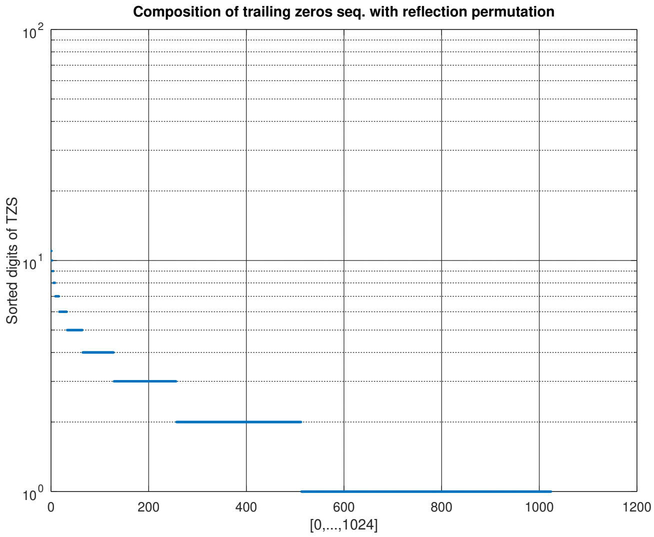

Figure 2: The composite sequence of the TZS over all reflected integers

One can then use directly this result to extract an integer histogram of all unique digits in any sub-sequence of TZS over any Mersenne interval and which follows a scaling law of the form $2^{L-l_2(i)-1}$ in accord with the tree structure also shown in figure 1.

Furthermore, it can be inductively verified that for any maximal interval $s(L)$ the following relation is always satisfied:

$$l_2(x) + (t \circ r)(x) = (l_2 \circ r)(x) + t(x) = L_{max} \quad (7)$$

where $L_{max} = Max(l_2(x))$.

Given the action of the reflector it is also possible to introduce an arithmetic decomposition of every palindrome's integer value of even order as:

$$\mathcal{P}_{2k}(|w|, L) = \mathcal{R}_k(|w|) + |w|2^k \quad (8)$$

The expression in (8) utilizes the involuntary nature of the reflector so as to get a sorted sequence of all possible even palindromes in any S_L. This is simply the result of taking two copies of any S_L matrix and perform a horizontal concatenation of the first copy with an up-down flip of the second copy, something easy to realize in an array language like Matlab or Octave.

Getting back the corresponding integer values will always result in a sorted sequence since the single application of $\mathcal{R}_L$ is equivalent to a permutation but the addition of all original bits above 2^L guarantees this being a sorted sequence.

Let then, $\Delta\mathcal{P}_L$ denote the discrete differences of the sorted sequence in (8) being again decomposed as:

$$\Delta\mathcal{P}_L(|w|, L) = \mathcal{P}_L(|w|, L) - \mathcal{P}_L(|w| - 1, L) = \Delta\mathcal{R}_L(|w|) + 2^L \quad (9)$$

where now

$$\Delta\mathcal{R}_L(|w|) = \mathcal{R}_L(|w|) - \mathcal{R}_L(|w| - 1) \quad (10)$$

It is then possible to prove the below proposition

for all $|w| \in \mathcal{M}(k)$ and their expansions $w \in S_k$ for any k it holds that
$\Delta\mathcal{P}_{2k}(|w|) = 3(2^{k-t(|w|)-1})$
where $t(|w|)$ the trailing zeros sequence.

The proof can be given with the aid of elementary curry-less binary addition performed on an unbounded or bi-infinite tape of a TM restricted in the case of a successor function $\mathfrak{s}(|w|) = |w| + 1$.

Indeed, adding a single bit at the lower power of any expansion only requires two rules. Assuming any integer n coded in binary with powers of two from left to right, these rules are:

- if n is even, the head writes '1' in the present position and stops.
- if n is odd, the head moves to the left replacing all 1s with 0s until it reaches the first 0 position where it writes a '1' and stops.

Next, consider the case of a bi-infinite tape with the first digit situated at a central cell with the whole pattern reflected as if by a mirror in the middle. One can alwasy assume a machine with two heads working in opposite directions but following the same pair of rules.

There can only be transitions from even to odd or from odd to even numbers. In both cases, any alterations in the first block of digits will not affect the next blocks so that they cannot contribute to the discrete differences of the resulting sequence of palindromes.

In the first, even-to-odd case, assume an arbitrarily large all 0s block reflected across the middle point. Then the transition will be of the form:

- $a_0 \ \cdots 1\ 0\ \cdots 0|0 \cdots 0\ 1 \cdots\ a_{2k}$
- $a_0 \ \cdots 1\ 0\ \cdots 1|1 \cdots 0\ 1 \cdots\ a_{2k}$

Since the whole palindrome is now a new pattern with an expansion of double length, the newly added 1s in the middle will correspond to a pair of new powers $\{a, 2a\}$ with $a = 2^{k-1}$. Then inevitably, the difference between the new integer advanced by one will have to be $3a = 3\ 2^{k-1}$.

In the second, odd-to-even case, assume again an arbitrarily large all 1s block in which case any transition will be of the form

- $a_0 \ \cdots 0\ 1\ \cdots 1|1 \cdots 1\ 0 \cdots\ a_{2k}$
- $a_0 \ \cdots 1\ 0\ \cdots 0|0 \cdots 0\ 1 \cdots\ a_{2k}$

By a similar argument as before, the new 0s block must contain a number of $2t(|w|)$ 0s. With both blocks shifted by the same amount of $a = 2^{k-t(|w|)-1}$ any difference becomes

$(2^{2t+1} + 1)a - 2(2^{2t} - 1)a = 3\ 2^{k-t(|w|)-1}$

Consequently, we also obtain the following computationally useful results.

A. The sequence of valuations of palindromic words over any $s(k)$ interval is given by the sequence of partial summands

$\mathcal{P}_{2k}(i) = 2^{k-1} 3 \sum_{i=1}^{2^k} 2^{-t(i)}$,

B. The sequence of reflectors over any $s(k)$ interval is given as

$\mathcal{R}_k(i) = 2^k \left(3 \sum_{i=1}^{2^k} 2^{-t(i)-1} - 1\right)$

In the light of the relation (7) the previous can be further simplified as:

$$\mathcal{P}_{2k}(i) = \frac{3}{2} \sum_{i=1}^{2^k} 2^{-(l_2(r(i))} \tag{11}$$

$$\mathcal{R}_k(i) = \frac{3}{2} \sum_{i=1}^{2^k} 2^{-(l_2(r(i))} - 2^k \tag{12}$$

One immediately notices here the presence of the magic factor $3/2^{t(x)}$ in the total expression of the corollary 1 for $\Delta\mathcal{P}$. This is then used to redefine the dynamics of the modified CH in (3) in a particular way revealing an interaction between two 'mirror' worlds.

4 A hidden mirror in the CH dynamics

It is obvious from direct comparison of the expression of the map in (3) and the result in proposition 2 that one should be able to make a direct substitution as

$$x_{n+1} = \left(\frac{\Delta\mathcal{P}_{l_{max}}(x_n)}{2^{l_{max}-1}}\right) x_n + 1 \tag{13}$$

where now the coefficient numerator in (10) is to be interpreted as:

$$\Delta\mathcal{P}_{l_{max}}(x_n) = \mathcal{P}_{l_{max}}(x_n) - \mathcal{P}_{l_{max}}(x_n - 1) = \Delta\mathcal{R}_{l_{max}}(x_n) + 2^{l_{max}} \tag{14}$$

In order for the substitution to make sense it has to be assumed that each x_n is taken inside an interval $s(l_2(x_n))$ which varies.

For this reason it is necessary to use a varying maximal length for the definition of the reflector as $l_{max} = Max(l_2(x))$ (practically $2^{l_{max}}$ is equivalent to the use of standard libraries like *nextpow2*). This becomes necessary due to the fact that there is a hierarchy of different $\mathcal{R}$ sequences or reflective permutation groups across different intervals $s(L)$. On the other hand, the original conjecture is equivalent to the existence of an upper bound for all such intervals.

The dynamics in (13) appears now as the result of an interaction of the original variable with the 'slope' formed by a discrete derivative over reflections. To further understand this version of the original dynamics it is necessary to find a reduction of the discrete difference in some more fundamental sequences like those introduced in the previous section.

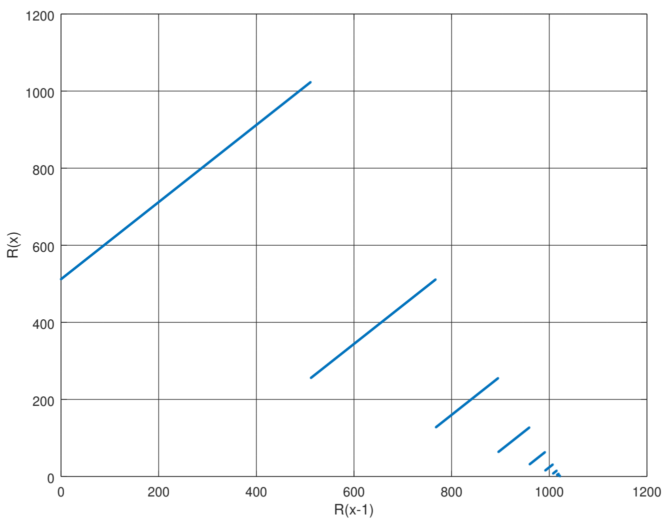

Figure 3: Global map for the reflection sequences

The simplest way is to utilize a global map for the pair $\{\mathcal{R}(x), \mathcal{R}(x-1)\}$. This is shown in figure 3, where a scaling law appears to govern the piece-wise linear dependence over successive intervals.

An arithmetic formula can be found inductively across the hierarchy of intervals utilizing the particular scaling which follows a similar pattern with that of the sorted TZS in figure 2.

It is then possible to prove inductively for the map of figure 3, the arithmetic interpolant

$$\mathcal{R}_L(x) = \mathcal{R}_L(x-1) - 2^{l_{max}} + 3 \times 2^{l(x)-1}$$

where now

$$\mathfrak{l}(x) = l_2(2^{l_{max}} - x - 1)$$

It should be noticed that the argument in $\mathfrak{g}$ performs a kind of parity reflection over any interval due to the arithmetic equivalent of the 2-complement defined as $NOT(x, L) = 2^L - 1 - x$.

Then the original reflector difference reduces to elementary sequences as:

$$\Delta\mathcal{R}_{l_{max}} = 2^{l_{max}} \left(3 \times 2^{\mathfrak{l}(x) - l_{max} - 1} - 1\right) \quad (15)$$

Substitution in (13) using (14) then results in:

$$x_{n+1} = 2\left(\frac{\Delta\mathcal{R}_{l_{max}}(x_n)}{2^{l_{max}}} + 1\right)x_n + 1 = 2^{\mathfrak{l}(x_n) - l_{max}}(3x_n) + 1 \quad (16)$$

What is actually gained in (16) is the expression of the same dynamics as in (3) but this time avoiding the difficulty of the fractal structure of the TZS sequence.

On the other hand, in the light of relation (7) it is also possible to write (3) as:

$$x_{n+1} = 2^{l_{max} - l_2(\mathcal{R}(x_n))}(3x_n) + 1 \quad (17)$$

The expression in (17) again emphasizes the role of reflections in the overall dynamics evident in the antagonism between the effective lengths of two mirror images in the exponent of (17).

Actually, the two expressions have now come full circle since the parity reflection in $\mathfrak{l}$ is just another identity in disguise or $\mathfrak{l}(x) - (l_2 \circ r)(x) = 2l_{max}$ on the exchange of parity reflections with the index binary reflections.

This alone is not sufficient to explain the mystery of the conjectured global convergence yet another alternative is offered in the last section which may be fruitful for further investigation in juxtaposition with the type of 'mirror' image interactions presented.

4.1 Convergence as block decimation

From the structure of (3) it is evident that any final convergence to the fixed point of the dynamics will take place as soon as the trajectory will reach a pure power of two.

In the light of the equivalent RL representation introduced in section 3.2, this can be phrased as a reduction of the number of blocks since any such is always of the form $\{\mathbf{0}^{c_0}, \mathbf{1}, \mathbf{0}^{c_2}\}$ for any $l_{max} = c_0 + c_2 + 1$.

The originally conjectured global convergence must then be equivalent to a higher probability of a falling number of blocks leading to what could be termed a *'block decimation'* effect although it is not stepwise homogeneous. A possible strategy for proving the original conjecture could then start with a proper definition of such a probability.

An effective measure of such a type of binary complexity index can be given in terms of the number of RL coefficients which may be termed here as the *RL Dimension* or RLD for brevity.

The particular type of RLD sequences per $s(L)$ interval also have a fractal character and can be found via induction over the hierarchy to satisfy a scaling law given by the standard list concatenation scheme.

$$rld(i+1) \leftarrow [rld(i), \mathcal{R}(rld(i)) + \sigma_i], rld(0) = 1$$

where now $\sigma_i = 1, i = 0, \cdots L - 1$ and $\sigma_L = 0$ while the reflection operator inverts the list index order at every step.

A natural property of any such sequence is its invariance under the reflection group over the indices themselves or simply $rld(\mathcal{R}(n)) = rld(n), n \in s(L)$ since reflection over fixed length binary expansions of each index cannot alter the number of the corresponding RL coefficients.

An example of such a fractal sequence can be seen in figure 4(a) over $s(10)$ while in figure 4(b) its distribution is compared against the standard binomial distribution of the 'Digit-Sum' sequence.

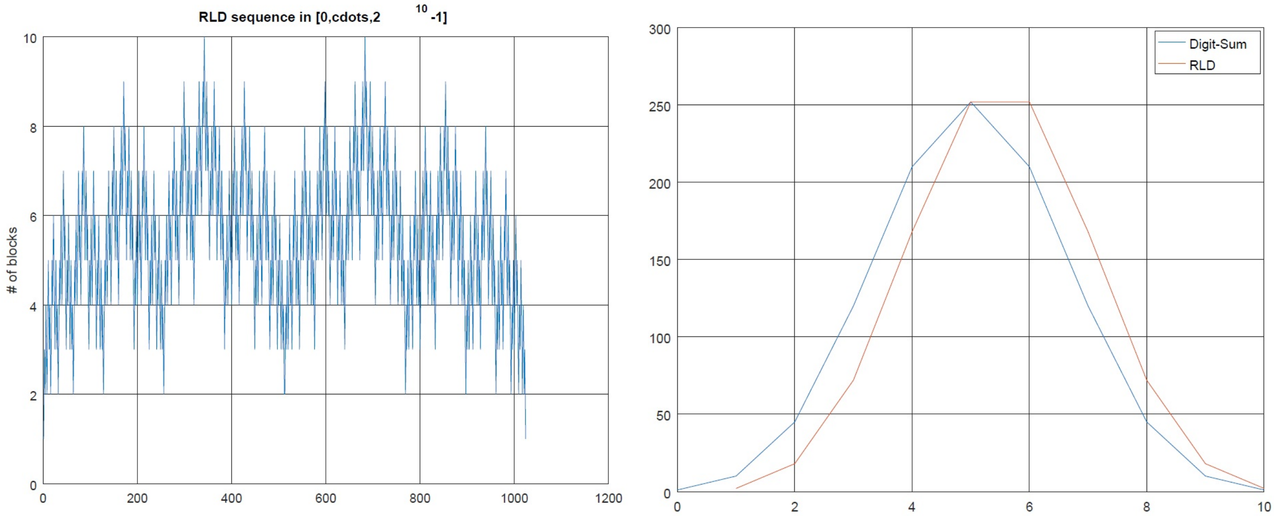

Figure 4: (a) Example of an RLD sequence on $s(10)$ and (b) Distributions for 'Digit-Sum' and RLD on $s(10)$

The simplest way to utilize these sequences is to define a probability of any odd integer being mapped to another even one via the '$3x + 1$' map of smaller or larger RLD. Given a long list of RLD values, this is straightforwardly written as

$$p_{<,>} = \frac{1}{\mu_L(rld)} \sum_{x=2k+1, x\in s(L)} sign\left[rld(3x+1) - rld(x)\right] \tag{18}$$

where x runs in all odd values in $s(L)$ and $\mu_L(rld)$ is an appropriate normalization measure over all 2^L values of RLD.

A preferable index for asserting any increase or decrease in the complexity of the resulting patterns can then be given by the probability mass ratio $p_</p_>$ which avoids normalization.

Additionally, the previous section finding suggests a correlation of such an index with the 'interaction' between mirror images of the expansions of the x_n variable via the quantity:

$$\delta_R(x) = l_2(3x+1) - l_2(\mathcal{R}(3x+1)) \tag{19}$$

The associated probabilities

$$q_{<,>} = \frac{1}{\mu_L} \sum_{x=2k+1, x\in s(L)} sign(\delta_R)(x) \tag{20}$$

allow defining another mass ratio as $q_</q_>$.

The new ratio can be used to check whether there is an increase or decrease in the resulting effective expansion length between these images. A decreasing ratio could be associated with a possible increase in the presence of large binary shifts thus diminishing the number of blocks to be eliminated.

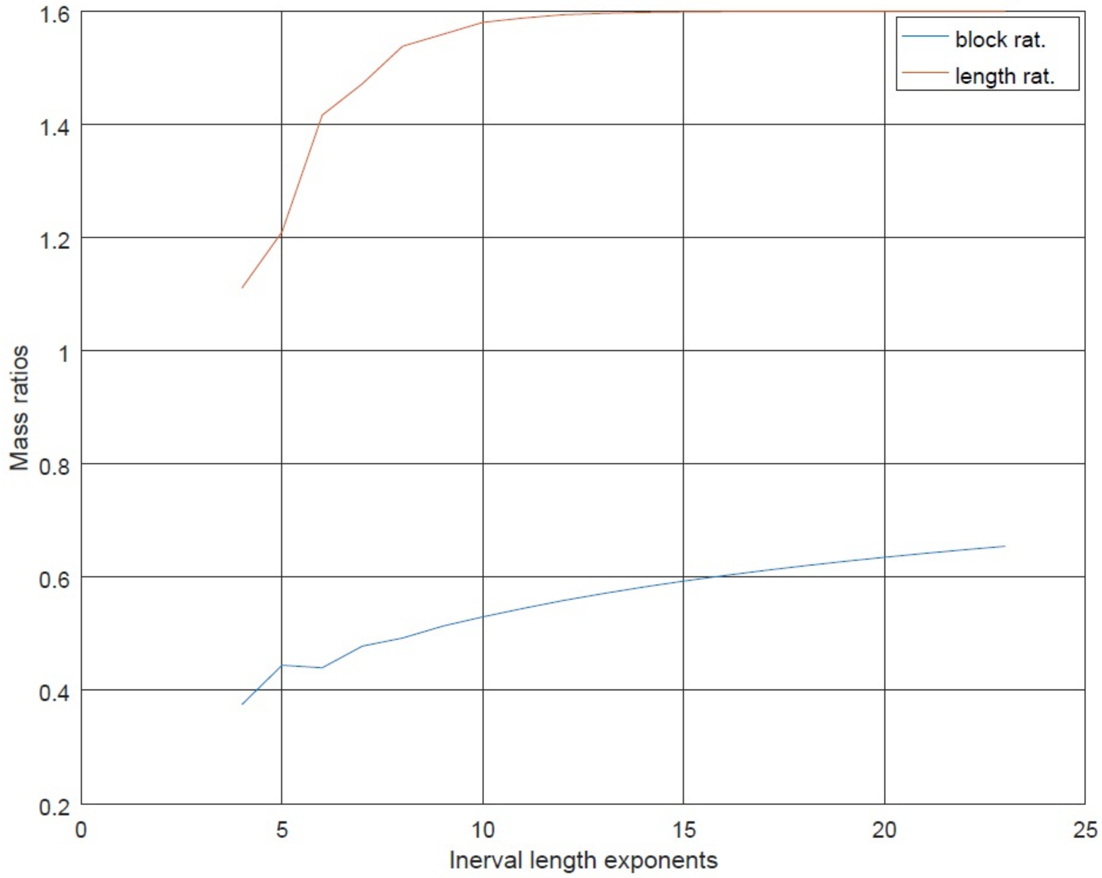

Figure 5: Combined probability mass ratios.

Indeed, numerical evidence is in favor of this assumption. Results for a set of intervals from $s(4)$ up to $s(24)$ are shown in figure 5 for both mass ratios. Interestingly, the lengths ratio appears saturated soon after the tenth power of two while the block ratio increases almost constantly in a log scale.

5 Discussion and Conclusions

A methodology for combinatorics of automata was introduced which may offer certain advantages regarding the extraction of scaling maps and recursive relations over lists as representatives of properties of fixed length binary patterns.

The particular application in the case of the Collatz-Hailstone dynamics was based on a coincidence after reformulating the original problem in a way that naturally incorporated the original branching condition via the use of a number theoretic function known as the trailing zeros sequence (TZS) otherwise known as the 2-adic valuation of the integers.

When comparing the new form with the sequence of discrete differences of palindromes defined via the action of the reflection group on a hierarchy of exponential intervals or closures over the integers, a deeper relation was recognised and further analysed.

It was revealed that internal reflections of the binary forms hidden behind the production of Hailstone sequences play a role not yet well understood. From a physicist's perspective it is tempting to think of this dynamics as a bistable potential with a middle barrier separating two mirror worlds perhaps amenable to noisy perturbations. Furthering this treatment is reserved for a future report.

Additionally, there are still unexplored issues regarding the multi-valued character of the TZS. As a matter of fact, the particular symbolic substitution used in (13) of section 4 could be generalized by simply allowing the index of the palindromic word or its internally contained reflection to be associated with any integer pre-image of the TZS giving the same value say as $\mathcal{R}(x_n) \to \mathcal{R}(x_n) \pm \sigma_n$ where r a random variable restricted each time to the same level of the TZS.

This brings about an interesting association of the natural tree structure of the TZS with a well known mechanical analog of the so called, *'Quincunx'* or *'Galton Machine'* [**?**] One can think of the additional random variable σ_n as the result of a falling ball across the quincunx board made out of integer spacings with its associated height variable thus recreating the exact same values of the TZS.

This leads to the amazing observation that despite σ_n being binomially distributed the Hailstone sequences would remain absolutely insensitive and hence the dynamics of this type would be an invariant of such a perturbation!

It is an ambitious project to carry over similar generalizations that overcomes the scope of the present short report which was based on a rather trivial original observation yet it was laid here in the hope that it may be of aid in future attempts towards a formal proof of the original conjecture by Collatz.

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