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The Structure of Caustics Formed by Reflection by Curves and Surfaces

Josh Pearson[1,*]

[1]University of Portsmouth, School of Mathematics and Physics, Portsmouth PO1 2HE, UK

[*]Corresponding author (Email: joshpearson910@gmail.com)

Abstract - In this paper we define a method for finding and plotting the caustics formed by reflection by curves. We then explicitly define the focal radius, ρ, and use it to make conclusions about the caustics of circles and ellipses. Some observations are made of the caustic's cusps before shifting our attention to surfaces. There, a similar definition is made before showing some of the possible caustics of spheres and ellipsoids. We finish with a brief discussion on a potential analogue to umbilic points.

Keywords - Caustic; Plane curve; Surface; Focal set; Singularities; Umbilics

1 Introduction

A caustic (catacaustic) is the envelope of light rays which have been reflected by a curve or surface from a light source (known as a radiant point). The caustic itself is a curve or surface to which each light ray is tangent. The aim of this paper is to study the behaviour of caustics and their underlying structure. Although there have previously been derivations for the caustics of curves [1], this project aims to provide a more general study in terms of curvatures and produce a similar derivation for the caustics of surfaces. With these results, the hope is that deeper analysis may be conducted into the behaviour of these caustics. Starting with curves, we begin by outlining the initial method used to find and plot the relevant caustics.

2 Caustics of Curves

We first assume the radiant point is fixed at the origin - this will be retained throughout the paper. A change in the relative position of the radiant point can be achieved by a shift in the curve. Each point on the parameterized curve $\boldsymbol{\gamma}$ will have a unit tangent vector $\hat{\mathbf{T}}$ and unit normal vector $\hat{\mathbf{N}}$. Every ray from the radiant point will be reflected from a point on $\boldsymbol{\gamma}$ such that the angle of incidence w.r.t $\hat{\mathbf{N}}$ is equal to the angle of reflection; we call this direction $\mathbf{R}$. This can be found from the Householder Transformation [2] applied to $\boldsymbol{\gamma}$ such that $\mathbf{R} = \boldsymbol{\gamma} - 2(\boldsymbol{\gamma} \cdot \hat{\mathbf{N}})\hat{\mathbf{N}}$. Points at which neighbouring rays intersect (focal points) will be a multiple ρ of this direction. Taking inspiration from the normal map seen in [3], a similar map may be created to describe a curve's caustic:

$$M : (s, \rho) \to (x, y) = \boldsymbol{\gamma}(s) + \rho\mathbf{R}(s)$$

We know M to be defined for all ρ and s but there are special values of ρ for which M is no longer invertible - the "focal radius". Hence, we can find where the Jacobian determinant $|DM|$ is equal to zero and solve this for ρ. With this we may then plot the corresponding curve and caustic.

Example 1: Take the unit circle with radiant point on its rim - this can be accomplished by shifting the curve by (1,0) to give $\boldsymbol{\gamma} = (1 + \cos\theta, \sin\theta)$. We can now proceed with the method outlined above:

$$\begin{aligned}\mathbf{R} &= \gamma - 2\left(\gamma \cdot \hat{\mathbf{N}}\right)\hat{\mathbf{N}} \\ &= \left(-2\cos^2\theta - \cos\theta + 1, -\sin\theta - 2\sin\theta\cos\theta\right)\end{aligned}$$

Now with $M = \boldsymbol{\gamma} + \rho\mathbf{R}$ we find

$$\begin{aligned}|DM| &= \begin{vmatrix} \frac{\partial M}{\partial \rho} \\ \frac{\partial M}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \left(-2\cos^2\theta - \cos\theta + 1, -\sin\theta - 2\sin\theta\cos\theta\right) \\ (-\sin\theta, \cos\theta) + \rho(4\sin\theta\cos\theta + \sin\theta, -\cos\theta - 2\cos 2\theta) \end{vmatrix} \\ &= \begin{vmatrix} \left(-2\cos^2\theta - \cos\theta + 1, -\sin\theta - 2\sin\theta\cos\theta\right) \\ (-\sin\theta + \rho(4\sin\theta\cos\theta + \sin\theta), \cos\theta + \rho(-\cos\theta - 2\cos 2\theta)) \end{vmatrix}.\end{aligned}$$

Using the definition of the determinant, after some rearranging, we are left with the following :

$$|DM| = (-1 + 3\rho)(1 + \cos\theta) = 0$$

in which when $\rho = \frac{1}{3}$ we see the reflection map is singular. Using this we can now plot the curve and its caustic as shown in Figure 1:

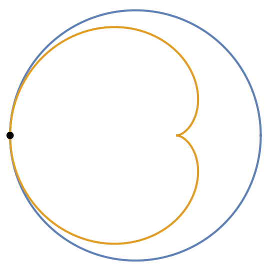


Figure 1: Caustic of unit circle with radiant point on its rim

2.1 Analytic Expression for ρ

The method previously described can be thought of as a "brute force" method - for a given curve we set out to find ρ before applying it to our map. If we had an exact expression for ρ, this process would be far less computationally strenuous, whilst also leading to greater understanding and more elaborate analysis. By first assuming the curve is arc length parameterized (such that $\frac{d\gamma}{ds} = \hat{\mathbf{T}}$) we can attempt to find such a solution. First recall

$$|DM| = \begin{vmatrix} \frac{\partial M}{\partial \rho} \\ \frac{\partial M}{\partial s} \end{vmatrix} = \begin{vmatrix} \mathbf{R} \\ \hat{\mathbf{T}} + \rho\dot{\mathbf{R}} \end{vmatrix} = 0, \ \dot{\mathbf{R}} = \frac{d\mathbf{R}}{ds}.$$

Using the Frenet-Serret equations [4] each vector can be split into the tangential and normal components of γ:

$$
\begin{aligned}
|DM| &= \begin{vmatrix} \left(\gamma\cdot\hat{\mathbf{T}}\right)\hat{\mathbf{T}} - \left(\gamma\cdot\hat{\mathbf{N}}\right)\hat{\mathbf{N}} \\ \hat{\mathbf{T}} + \rho\left(\hat{\mathbf{T}} - 2\left(\left(\gamma\cdot\hat{\mathbf{N}}\right)\left(-\kappa\hat{\mathbf{T}}\right) + \left(\hat{\mathbf{T}}\cdot\hat{\mathbf{N}} + \gamma\cdot\left(-\kappa\hat{\mathbf{T}}\right)\right)\hat{\mathbf{N}}\right)\right) \end{vmatrix} \\
&= \begin{vmatrix} \left(\gamma\cdot\hat{\mathbf{T}}\right)\hat{\mathbf{T}} - \left(\gamma\cdot\hat{\mathbf{N}}\right)\hat{\mathbf{N}} \\ \hat{\mathbf{T}} + \rho\left(\hat{\mathbf{T}} + 2\kappa\left(\left(\gamma\cdot\hat{\mathbf{N}}\right)\hat{\mathbf{T}} + \left(\gamma\cdot\hat{\mathbf{T}}\right)\hat{\mathbf{N}}\right)\right) \end{vmatrix} \\
&= \begin{vmatrix} \left(\gamma\cdot\hat{\mathbf{T}}\right)\hat{\mathbf{T}} - \left(\gamma\cdot\hat{\mathbf{N}}\right)\hat{\mathbf{N}} \\ \left(1 + \rho\left(1 + 2\kappa\left(\gamma\cdot\hat{\mathbf{N}}\right)\right)\right)\hat{\mathbf{T}} + 2\rho\kappa\left(\gamma\cdot\hat{\mathbf{T}}\right)\hat{\mathbf{N}} \end{vmatrix}.
\end{aligned}
$$

The determinant of the product is the product of the determinants and so

$$
|DM| = \begin{vmatrix} \left(\gamma\cdot\hat{\mathbf{T}}\right) & -\left(\gamma\cdot\hat{\mathbf{N}}\right) \\ \left(1 + \rho\left(1 + 2\kappa\left(\gamma\cdot\hat{\mathbf{N}}\right)\right)\right) & 2\rho\kappa\left(\gamma\cdot\hat{\mathbf{T}}\right) \end{vmatrix} \begin{vmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}} \end{vmatrix}.
$$

As the basis $\{\hat{\mathbf{T}}, \hat{\mathbf{N}}\}$ is orthonormal we know $\det\left(\hat{\mathbf{T}}, \hat{\mathbf{N}}\right) = \pm 1$ [4]. Hence, for the above equation to equal zero we must have

$$
\begin{vmatrix} \left(\gamma\cdot\hat{\mathbf{T}}\right) & -\left(\gamma\cdot\hat{\mathbf{N}}\right) \\ \left(1 + \rho\left(1 + 2\kappa\left(\gamma\cdot\hat{\mathbf{N}}\right)\right)\right) & 2\rho\kappa\left(\gamma\cdot\hat{\mathbf{T}}\right) \end{vmatrix} = 0.
$$

After some simplification we may conclude that

$$
\rho = \frac{-(\gamma\cdot\hat{\mathbf{N}})}{2\kappa\,||\gamma||^2 + (\gamma\cdot\hat{\mathbf{N}})} \tag{1}
$$

independent of parameterization. To emphasise the utility of this formula we can attempt the same example again, now directly calculating ρ from the beginning.

Example 2:

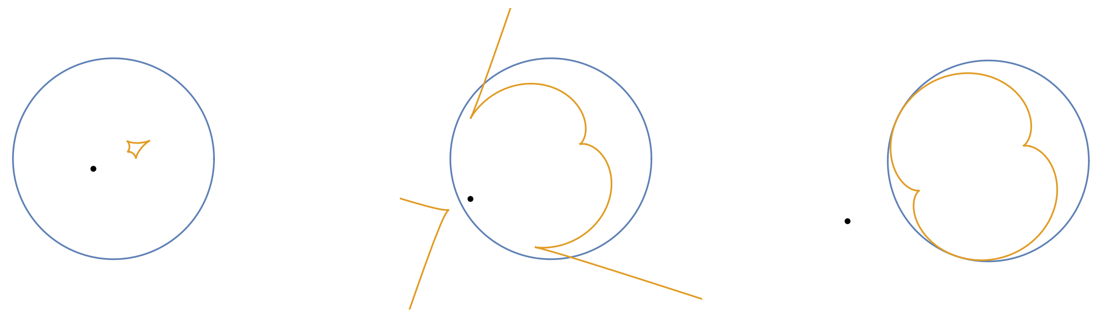
$$
\begin{aligned}
\gamma &= (1 + \cos\theta, \sin\theta) \\
\implies \hat{\mathbf{N}} &= (-\cos\theta, -\sin\theta) \\
\implies \left(\gamma\cdot\hat{\mathbf{N}}\right) &= -\cos\theta - 1, \quad ||\gamma||^2 = 2\cos\theta + 2
\end{aligned}
$$

Using this along with $\kappa = 1$ we see

$$
\rho = \frac{\cos\theta + 1}{4\cos\theta + 4 - \cos\theta - 1} = \frac{1}{3}.
$$

The same result - now far easier to calculate. Throughout this paper more uses of this expression can be seen, particularly in Figures 2,4 and 6. With this in place we may go on to make further conclusions about the caustics of curves - in particular those of circles and ellipses.

2.2 Compact Regions of Circles



$A < \frac{1}{2} \implies$ compact $\qquad \frac{1}{2} \leq A < 1 \implies$ not compact $\qquad A \geq 1 \implies$ compact

Figure 2: Caustics illustrating compact regions of the unit circle

In the context of this paper, a "compact region" will be defined as the region in the plane relative to the curve in which a radiant point produces a compact (closed and bounded) caustic. Importantly, these compact regions may be disjoint as seen in Figure 2. Equally, stating a point or set of points in the plane "produces a compact caustic" will be abbreviated to stating the point is "compact" for the sake of brevity. For a caustic to be compact, ρ must remain finite across the entirety of its domain. Now with an explicit function for ρ we aim to distinguish between compact and non-compact regions - beginning with the unit circle. All unit circles can be described by $\boldsymbol{\gamma} = (\alpha + \cos\theta, \beta + \sin\theta)$, $\theta \in [0, 2\pi]$ where $\alpha, \beta \in \mathbb{R}$. Using equation (1) we see

$$\rho = \frac{1 + \alpha\cos\theta + \beta\sin\theta}{1 + 2\left(\alpha^2 + \beta^2\right) + 3\left(\alpha\cos\theta + \beta\sin\theta\right)}. \tag{2}$$

Hence, the resulting caustics will be compact if the denominator is never zero for all $\theta \in [0, 2\pi]$. Now, we can find the values of α and β where the curve's caustic will be compact:

$$\begin{aligned} & 1 + 2\left(\alpha^2 + \beta^2\right) + 3\left(\alpha\cos\theta + \beta\sin\theta\right) = 0 \\ \implies & 1 + 2\left(\alpha^2 + \beta^2\right) + 3\sqrt{\alpha^2 + \beta^2}\cos\left(\theta - \phi\right) = 0 \end{aligned}$$

where $\phi = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$ assuming $\alpha > 0$. Hence

$$\cos\left(\theta - \phi\right) = -\frac{1}{3}\left(\frac{1 + 2A^2}{A}\right), \quad A = \sqrt{\alpha^2 + \beta^2}.$$

As $\cos\left(\theta - \phi\right) \in [-1, 1]$, if RHS $\in [-1, 1]$ we can determine that the caustic will not be compact (as there will be points at which the denominator of (2) is zero). As $A \geq 0$, we know RHS≤ 0 and so if $\frac{1+2A^2}{A} \in [0, 3]$ the caustic is not compact:

$$\begin{aligned} \frac{1 + 2A^2}{A} \leq 3 \implies & 1 + 2A^2 \leq 3A \\ \implies & 2A^2 - 3A + 1 \leq 0 \\ \implies & \tfrac{1}{2} \leq A \leq 1. \end{aligned}$$

By checking the endpoints, $A = \frac{1}{2}$ and $A = 1$, we find that $A = 1$ is compact. This is demonstrated in Examples 1 and 2 in which $\alpha = 1$ and $\beta = 0$ ($A = 1$) where it was found $\rho = \frac{1}{3}$. Therefore, we can state that caustics of the unit circle are not compact for

$$\tfrac{1}{2} \leq A < 1 \text{ where } A = \sqrt{\alpha^2 + \beta^2} \tag{3}$$

2.3 Compact Regions of Ellipses

Having found the compact regions of the unit circle we now aim to do the same for ellipses. One question we can answer analytically is at what eccentricity does the centre cease to be compact? As the ellipse's centre is at the origin, all relevant ellipses can be described by the curve $\boldsymbol{\gamma} = (\mu\cos\theta, \lambda\sin\theta)$, $\theta \in [0, 2\pi]$ where $\mu, \lambda \in \mathbb{R}$. By fixing $\lambda = 1$ we can vary μ to find where the origin is no longer compact. Hence, we will be looking at $\boldsymbol{\gamma} = (\mu\cos\theta, \sin\theta)$. Once again using equation (1) we see

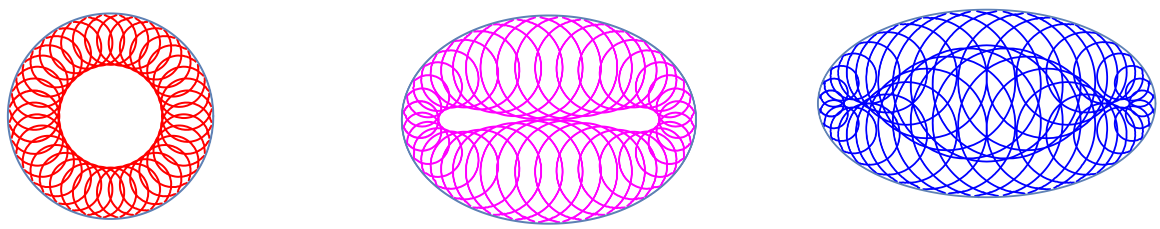
$$\rho = \frac{2\left(\cos^2\theta + \mu^2\sin^2\theta\right)}{1 + \mu^2 + 3\left(\mu^2 - 1\right)\cos 2\theta}.$$

Looking ahead to Figure 3 we see that increasing μ stretches the ellipse horizontally and collapses the compact region in the $\theta = \frac{\pi}{2}$ direction. Hence, letting $\theta = \frac{\pi}{2}$ yields

$$\rho = \frac{\mu^2}{2 - \mu^2},$$

and so we can conclude the origin is no longer compact at $\mu = \sqrt{2}$ or an eccentricity of $e = \frac{1}{\sqrt{2}}$.

Although a similar ρ can be defined for all ellipses, centred at the origin or otherwise, it may instead be simpler to opt for a more graphical method when discussing compact regions. Let $\boldsymbol{\gamma} = (\alpha + \mu\cos\theta, \beta + \lambda\sin\theta)$ where $\alpha, \beta, \mu, \lambda \in \mathbb{R}$. We begin, once again, by fixing $\lambda = 1$ and varying the eccentricity through changes in μ. This means that for a given μ and θ there will be a set of α and β such that $\frac{1}{\rho_{ellipse}} = 0$. If we plot these sets for many values of θ it starts to become clear which regions will not produce compact caustics:



$\mu = 1$ $\quad$ $\mu = \sqrt{2}$ $\quad$ $\mu = 1.8$

Figure 3: Non-compact regions of ellipses highlighted in red, pink and blue respectively

One feature common for all ellipses is that the curve and its exterior will always be compact, as seen in the examples of Figure 3. The result on the left acts as a visual representation of our result in equation (3), Section 2.2. As μ increases the centre stops being compact - particularly at $\mu = \sqrt{2}$ confirming our previous result. Beyond this, the only regions interior to the curve which continue to be compact are those around the ellipse's foci. These regions get smaller as the eccentricity increases until only the foci themselves are compact. This aligns with the long known "Reflective Property" of an ellipse discovered as far back as 200BC by Apollonius [5]. Some examples of the resulting caustics from various ellipses are provided in Figure 4:

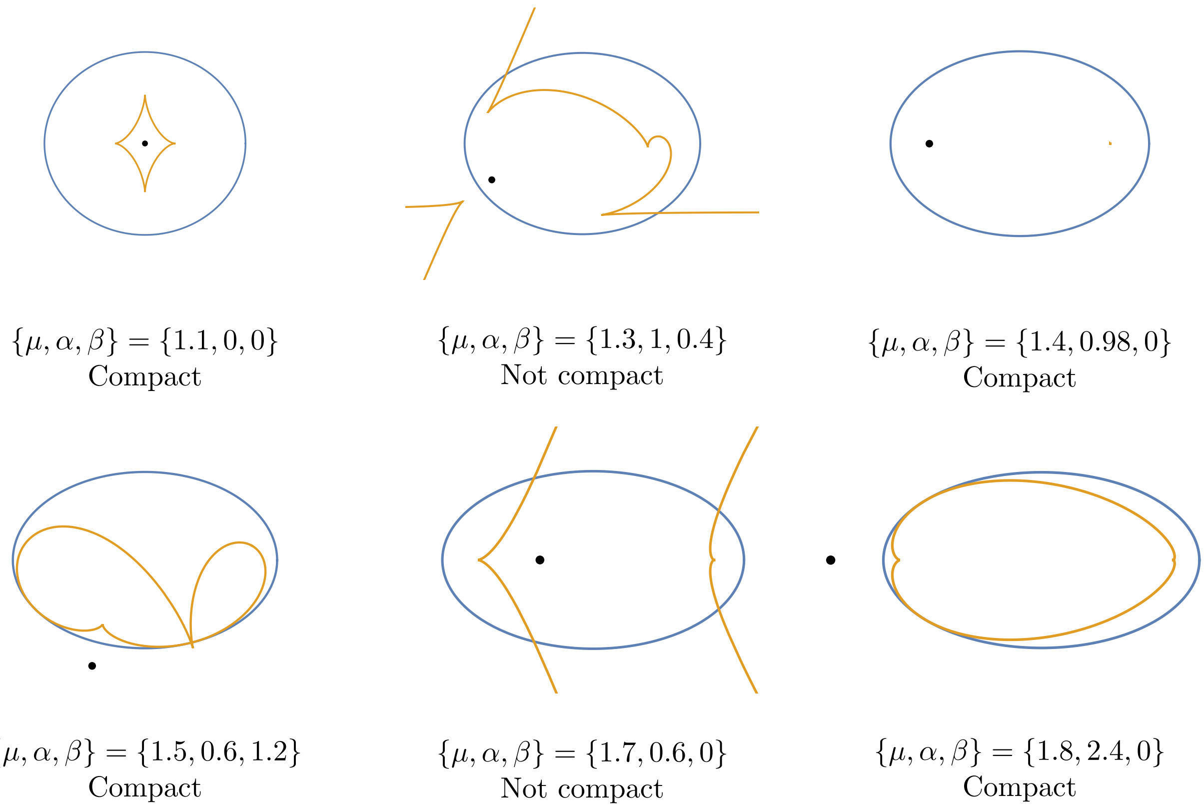


Figure 4: Caustics of various ellipses

2.4 Cusps

From all of the caustics previously shown one thing is common among almost all of them - cusps. As ρ is a function of s our map takes one input, s, and produces a point in the plane. Therefore, these cusps must occur when the derivative of our map, M, has an absolute value of zero - or equivalently when the tangent vector vanishes. In terms of our map, this can be written

$$\left| \dot{\gamma}(s) + \dot{\rho}(s)\mathbf{R}(s) + \rho(s)\dot{\mathbf{R}}(s) \right| = 0.$$

Although an equivalent expression was developed, describing the locations of cusps in terms of κ and its derivatives, it was found to be too long to mention in this paper. Instead, we shall go on to make some general observations about the behaviour of cusps. Firstly, for a radiant point in the interior of the curve, excluding foci where the caustic is a singular point, there appears to be four cusps. Although this is easier to see when our caustic is compact they nonetheless appear to be there for all caustics. This is seen in the fifth caustic in figure 4 in which 2 cusps are visible. However, taking a step back another 2 become visible, see Figure 5:

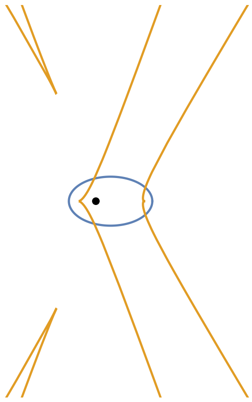


Figure 5: Caustic of ellipse expanded to show all 4 cusps

For a radiant point lying on the curve a single cusp forms. This has been seen for circles where the caustic takes the shape of a cardioid (Figure 1). In the case of the ellipse, this cardioid is warped and stretched depending on the location and eccentricity of the ellipse in question:

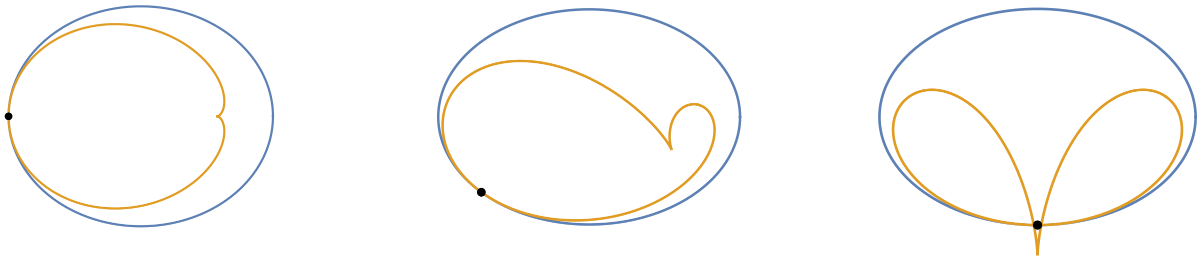


Figure 6: Caustics of ellipses with radiant point on the curve

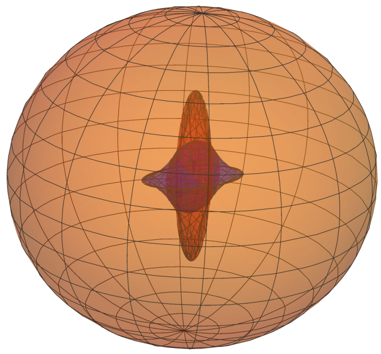
Finally, for radiant points outside the curve, 2 cusps are visible (Figure 2 and 4). Importantly, these results may not be conclusive for all curves - only the limited few we have studied. With more time, perhaps a more definitive solution could have been developed to describe the behaviour of cusps for all curves. Having discussed curves quite comprehensively it is time to shift our attention to the caustics of surfaces.

3 Caustics of Surfaces

In this section we shall attempt to provide a similar analysis for the caustics of surfaces. Hence, we shall redefine the initial "brute force" method and go on to define an expression for ρ within the context of surfaces. Instead of a curve $\boldsymbol{\gamma}$, we now have a surface $\boldsymbol{\sigma}(u, v)$ with unit normal $\hat{\mathbf{N}} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v|}$ and reflection map

$$M : (u, v, \rho) \to (x, y, z) = \boldsymbol{\sigma}(u, v) + \rho \mathbf{R}(u, v).$$

In this case $\mathbf{R} = \boldsymbol{\sigma} - 2(\boldsymbol{\sigma} \cdot \hat{\mathbf{N}})\hat{\mathbf{N}}$ by applying the Householder Transformation to $\boldsymbol{\sigma}$. As with curves, we solve $|DM| = 0$ for ρ with one key difference - there will now be 2 solutions. This is easily seen in the ellipsoid in Figure 7 in which 2 caustics are visible (one blue, one red) corresponding to the two solutions for ρ:



$$\boldsymbol{\sigma} = \{1.1\cos\phi\sin\theta, 1.05\sin\phi\sin\theta, \cos\theta\}$$

Figure 7: Caustics of an ellipsoid corresponding to ρ_1 and ρ_2

Now knowing that this method can be translated to surfaces, we focus our attention on finding ρ explicitly. With this in hand we hope to further our understanding of the caustics of surfaces.

3.1 Analytic Expression for ρ

As was the case previously, deriving an equation for ρ relies on us being able to split vectors into the tangential and normal components of $\boldsymbol{\sigma}$. However, we now have many pairs of potential tangents to consider - some being more useful than others. By choosing the orthonormal basis $\{\hat{\boldsymbol{\sigma}}_u, \hat{\boldsymbol{\sigma}}_v, \hat{\mathbf{N}}\}$, where $\hat{\boldsymbol{\sigma}}_u, \hat{\boldsymbol{\sigma}}_v$ are the surface's principal directions, we benefit from the knowledge that

$$\hat{\mathbf{N}}_u = -\kappa_1 \boldsymbol{\sigma}_u, \quad \hat{\mathbf{N}}_v = -\kappa_2 \boldsymbol{\sigma}_v$$

in which

$$\hat{\mathbf{N}}_u = \frac{\partial \hat{\mathbf{N}}}{\partial u}, \quad \boldsymbol{\sigma}_u = \frac{\partial \boldsymbol{\sigma}}{\partial u}, \quad \hat{\mathbf{N}}_v = \frac{\partial \hat{\mathbf{N}}}{\partial v}, \quad \boldsymbol{\sigma}_v = \frac{\partial \boldsymbol{\sigma}}{\partial v}$$

where κ_1 and κ_2 are the surface's principal curvatures [4]. Applying this we see

$$\begin{aligned}
|DM| = \begin{vmatrix} \frac{\partial M}{\partial \rho} \\ \frac{\partial M}{\partial u} \\ \frac{\partial M}{\partial v} \end{vmatrix} &= \begin{vmatrix} \mathbf{R} \\ \boldsymbol{\sigma}_u + \rho \mathbf{R}_u \\ \boldsymbol{\sigma}_v + \rho \mathbf{R}_v \end{vmatrix} \\
&= \begin{vmatrix} (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\sigma}}_u)\, \hat{\boldsymbol{\sigma}}_u + (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\sigma}}_v)\, \hat{\boldsymbol{\sigma}}_v - \left(\boldsymbol{\sigma} \cdot \hat{\mathbf{N}}\right) \hat{\mathbf{N}} \\ \boldsymbol{\sigma}_u + \rho \left(\boldsymbol{\sigma}_u - 2\left(\left(\boldsymbol{\sigma} \cdot \hat{\mathbf{N}}\right) \hat{\mathbf{N}}_u + \left(\boldsymbol{\sigma} \cdot \hat{\mathbf{N}}_u\right) \hat{\mathbf{N}}\right)\right) \\ \boldsymbol{\sigma}_v + \rho \left(\boldsymbol{\sigma}_v - 2\left(\left(\boldsymbol{\sigma} \cdot \hat{\mathbf{N}}\right) \hat{\mathbf{N}}_v + \left(\boldsymbol{\sigma} \cdot \hat{\mathbf{N}}_v\right) \hat{\mathbf{N}}\right)\right) \end{vmatrix} \\
&= \begin{vmatrix} (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\sigma}}_u)\, \hat{\boldsymbol{\sigma}}_u + (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\sigma}}_v)\, \hat{\boldsymbol{\sigma}}_v - \left(\boldsymbol{\sigma} \cdot \hat{\mathbf{N}}\right) \hat{\mathbf{N}} \\ ||\boldsymbol{\sigma}_u|| \left(\hat{\boldsymbol{\sigma}}_u + \rho \left(\hat{\boldsymbol{\sigma}}_u + 2\kappa_1 \left(\left(\boldsymbol{\sigma} \cdot \hat{\mathbf{N}}\right) \hat{\boldsymbol{\sigma}}_u + (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\sigma}}_u)\, \hat{\mathbf{N}}\right)\right)\right) \\ ||\boldsymbol{\sigma}_v|| \left(\hat{\boldsymbol{\sigma}}_v + \rho \left(\hat{\boldsymbol{\sigma}}_v + 2\kappa_2 \left(\left(\boldsymbol{\sigma} \cdot \hat{\mathbf{N}}\right) \hat{\boldsymbol{\sigma}}_v + (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\sigma}}_v)\, \hat{\mathbf{N}}\right)\right)\right) \end{vmatrix}
\end{aligned}$$

First notice that $||\boldsymbol{\sigma}_u||$ and $||\boldsymbol{\sigma}_v||$ can be factored out. Also, by choosing the orthonormal basis $\{\hat{\boldsymbol{\sigma}}_u, \hat{\boldsymbol{\sigma}}_v, \hat{\mathbf{N}}\}$, a similar method can be used from the previous derivation for ρ because this too has

$\det\left(\hat{\boldsymbol{\sigma}}_u, \hat{\boldsymbol{\sigma}}_v, \hat{\mathbf{N}}\right) = \pm 1$. Therefore, we must have

$$\begin{vmatrix} (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\sigma}}_u) & (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\sigma}}_v) & -\left(\boldsymbol{\sigma}\cdot\hat{\mathbf{N}}\right) \\ \left(1+\rho\left(1+2\kappa_1\left(\boldsymbol{\sigma}\cdot\hat{\mathbf{N}}\right)\right)\right) & 0 & 2\rho\kappa_1\left(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\sigma}}_u\right) \\ 0 & \left(1+\rho\left(1+2\kappa_2\left(\boldsymbol{\sigma}\cdot\hat{\mathbf{N}}\right)\right)\right) & 2\rho\kappa_2\left(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\sigma}}_v\right) \end{vmatrix} = 0.$$

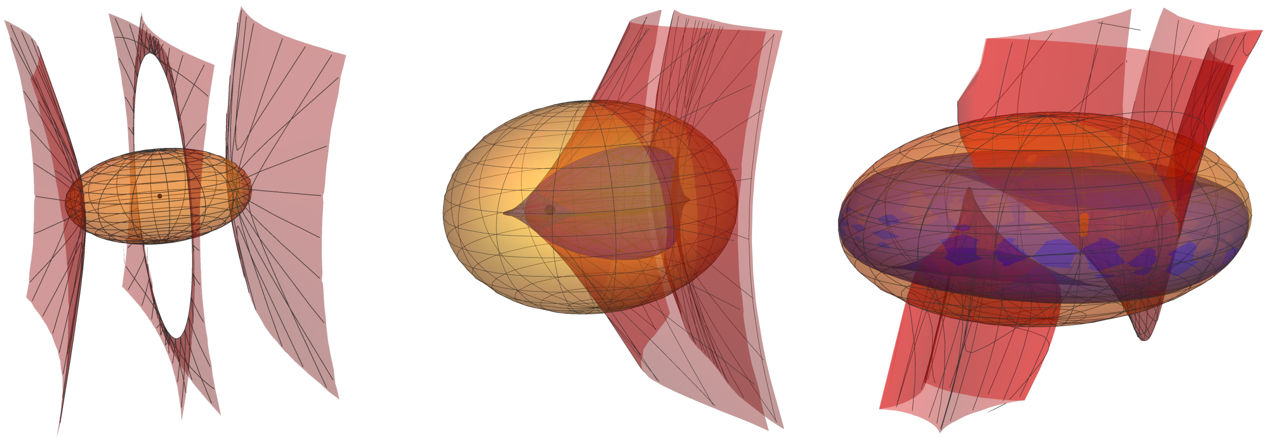
Letting $a = (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\sigma}}_u)$, $b = (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\sigma}}_v)$, $c = \left(\boldsymbol{\sigma}\cdot\hat{\mathbf{N}}\right)$ we find:

$$\rho = \frac{-c}{c + a^2\kappa_1 + b^2\kappa_2 + c^2(\kappa_1+\kappa_2) \pm \sqrt{a^4\kappa_1^2 + 2a^2\kappa_1\left(c^2\left(\kappa_1-\kappa_2\right) + b^2\kappa_2\right) + \left(c^2\left(\kappa_1-\kappa_2\right) - b^2\kappa_2\right)^2}} \tag{4}$$

or equivalently

$$\rho = \frac{-c}{c + 2H\left|\left|\boldsymbol{\sigma}\right|\right|^2 - (b^2\kappa_1 + a^2\kappa_2) \pm \sqrt{\left(2H\left|\left|\boldsymbol{\sigma}\right|\right|^2 - (b^2\kappa_1 + a^2\kappa_2)\right)^2 - 4c^2K\left|\left|\boldsymbol{\sigma}\right|\right|^2}} \tag{5}$$

in which H is the mean curvature, K is the Gaussian curvature and $\left(b^2\kappa_1 + a^2\kappa_2\right)$ is linked to the normal curvature κ_n. Some examples of the potential caustics found using this expression are seen in Figure 8.



$$\boldsymbol{\sigma} = \begin{pmatrix} 2\cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix} \qquad \boldsymbol{\sigma} = \begin{pmatrix} 0.4 + 1.4\cos\phi\sin\theta \\ 1.1\sin\phi\sin\theta \\ \cos\theta \end{pmatrix} \qquad \boldsymbol{\sigma} = \begin{pmatrix} 3 + 2\cos\phi\sin\theta \\ 1 + 1.3\sin\phi\sin\theta \\ 6 + \cos\theta \end{pmatrix}$$

Figure 8: Caustics of a variety of ellipsoids

Clearly there are significant differences between the caustics pictured above. In future, it may be interesting to more closely study the behaviour of these caustics - where they are compact, why in some cases they degenerate (left image of Figure 8 is missing a second caustic) and the interactions between the two caustics. Instead, for now, we shall briefly touch on the caustics of spheres. Due to their symmetry, an interesting solution arises. Using equation (4) above we let $\kappa = \kappa_1 = \kappa_2$ (as all points on the sphere have equal curvature) and after some simplification:

$$\rho_1 = \frac{-1}{1 + 2\kappa(\boldsymbol{\sigma}\cdot\hat{\mathbf{N}})},$$
$$\rho_2 = \frac{-(\boldsymbol{\sigma}\cdot\hat{\mathbf{N}})}{2\kappa\left|\left|\boldsymbol{\sigma}\right|\right|^2 + (\boldsymbol{\sigma}\cdot\hat{\mathbf{N}})}$$

with ρ_2 mimicking equation (1)'s result for curves. The caustics generated from ρ_2 look similar the those seen in circles, now rotated about the sphere, as seen in Figure 9:

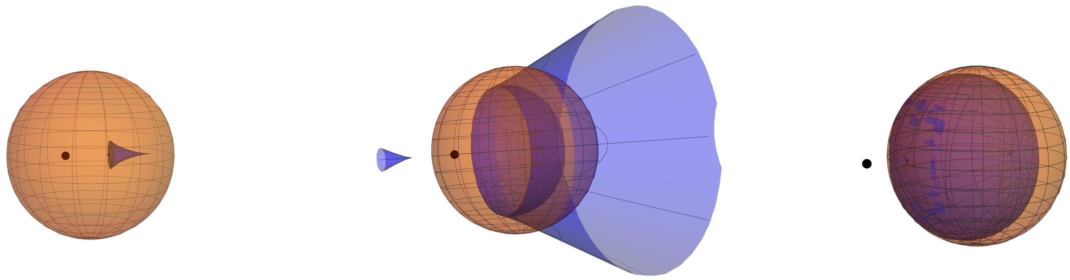


Figure 9: Caustics of the unit sphere similar to those seen in circles

with no caustic visible from ρ_1 as it degenerates into a pole intersecting the radiant point and the sphere's centre.

3.2 Focal Umbilics

The notion of umbilics already exists - points on a surface with equal normal curvature in all directions (hence $\kappa_1 = \kappa_2$). These umbilics give structure to a surface [6]. We propose the existence of "focal umbilics" being points in which our "focal radii" ρ_1 and ρ_2 are equal. Perhaps they too give structure to a surface, or rather its caustics. For the ellipsoid pictured in Figure 7 we can plot the two focal radii as seen below:

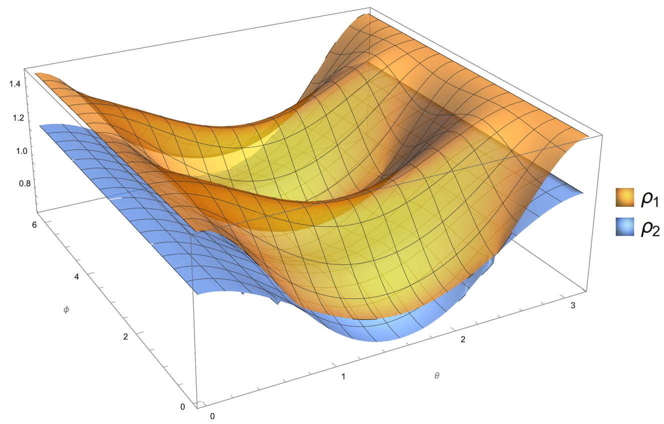


Figure 10: Focal radii of ellipsoid

Clearly there are points in which the focal radii are equal. This is seen in Figure 7 as points at which the two caustics touch. Alternatively, we can make a spherical plot with radius $(\rho_1 - \rho_2)$ (see Figure 11). Here we see four points approach the centre. These four points have zero radius meaning our focal radii are equal. Perhaps, like umbilics, these points have some control over the global behaviour of the caustics. This merits further investigation.

4 Conclusions

Through our investigation we have been able to study the caustics of both curves and surfaces. Beginning with curves, finding an expression for ρ greatly simplified our calculations and allowed for deeper analysis - particularly into compact regions and to a lesser extent cusps. As previously mentioned, with more time a more comprehensive study of cusps and their relevant properties could be achieved. Perhaps the number of cusps in each region is related to the number of cusps the curve itself has - a question for

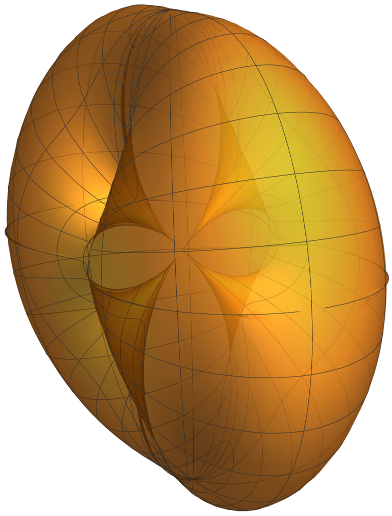


Figure 11: spherical plot of focal radii with $r = (\rho_1 - \rho_2)$

another time. A similar expression for ρ pertaining to surfaces was also developed however potentially less refined. With a deeper knowledge of differential geometry perhaps this equation could be simplified further. Our attention then shifts to these "focal umbilics". Although little was concluded, it was nonetheless an interesting idea to propose. Equally fascinating to explore would be ribs (an analogue to the cusps seen in curves). It may be found that they too have a rich and complex structure to be uncovered. All this is to say, despite the nature of the topic appearing simple, there is still much to be studied about caustics and their properties.

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