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Revisiting Some Leibnizian Concepts in Einstein's Gravitational Theory

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Abstract - We will discuss the analogies between principles and concepts established by G. Leibniz and their potential repercussions on Einstein's Theory of Gravitation. Our focus is on a formulation of Einstein's Equivalence Principle, specifically, the infinitesimal [strong] formulation of Einstein's principle. We will discuss Leibnizian concepts relevant to the technique of infinitesimal parallel transport of vectors, including the so-called "Schild ladder". To this end, we will address the Principle of Identity of Indiscernibles and other pertinent Leibnizian ideas, particularly those related to infinitesimals. We will point out that it is possible to establish connections between Leibniz's fundamental contributions and the framework of General Relativity. Although there are extensive debates in the literature on these topics, we identify new issues within General Relativity that deserve attention.

Keywords - General Relativity; Equivalence Principle; Schild's Ladder; Principle of Indiscernibles; Leibnizian Concepts.

1 Introduction

[1] The enormous relevance of Einstein's Theory of Gravitation, also known as the General Theory of Relativity (GTR), to Physics is indisputable. Moreover, more than a century since its inception, its history continues to reveal physical, mathematical, and epistemological questions that become the subject of fascinating new studies, and for this reason it is considered one of the most beautiful achievements of the human mind ([2], p.96). It is also well known that, within this theory, Einstein's Equivalence Principle played a prominent role in its formulation. Authors such as R. Penrose and S. Hawking ([3], p.62) have comparatively assessed the accuracy of GTR against the predictions of Quantum Field Theory, which had long been regarded as the physical theory with the most precise

[1]In this work, we shall consider Einstein's classical (non-quantum) theory of gravitation, in which the geometry of space-time is pseudo-Riemannian, torsion-free, and the covariant derivative of the metric vanishes ([1], p.4). Furthermore, our considerations regarding Leibniz shall adhere to the traditional formulations of his principles; our discussions shall be framed strictly within this context.

experimental outcomes. The accuracy of Quantum Field Theory was estimated at one part in 10^{+11}; however, in the aforementioned work, they estimated that GTR had been tested through pulsar observations and demonstrated an accuracy of one part in 10^{+14}. This led them to regard GTR as the most precise physical theory. In 2017 [4] and again in 2022 [5], new experiments confirmed, through different tests, that the [weak] Equivalence Principle (EP) showed no violations down to the 10^{-14} scale. It is also worth noting that other experiments on the *weak* version of the EP, involving antimatter ([6], p.716), confirmed the predictions of GTR. [2]; the space mission "NASA MESSENGER" ([7], p.1), which spent years investigating the ephemerids of the planet Mercury, likewise confirmed the [strong] EP.

Despite these experimental and observational results, it is well understood that, within the theoretical and epistemological context, the principles underpinning physical theories are invariably the subject of ongoing debate and controversy in various respects. Since the inception of GTR, such questions have arisen; let us recall some of them. In 1930, for instance, A. Eddington ([8], p.41) remarked that "the Principle of Equivalence has played a great part as a guide in the original building up of the generalized relativity theory; but now that we have reached the new view of the nature of the world it has become less necessary." There is also a well-known criticism of the EP frequently cited in the relevant literature, namely that found in the preface to the renowned book *Relativity: The General Theory* by J. L. Synge, published in 1960 ([9], p.IX), which contains a critique of the EP. However, here we prefer to recall what he later stated ([10], p.8):

> So, at the risk of boring those who look at Einstein's theory very much as I do, I shall say my say. This is a brief conducted tour along a route which deliberately avoids a pair of decaying and dangerous ruins, one named the Principle of Equivalence, and the other Mach's Principle. I have never understood the meaning of either.

We also find discussions pointing in the opposite direction. For example, Steven Weinberg ([11], pp. vii–viii), in his treatment of Einsteinian gravitation, adopted a different stance regarding the EP, emphasizing precisely the opposite of the authors mentioned above, as the Equivalence Principle is regarded as an essential component of his formulation. Weinberg asserted that "Riemannian geometry" appears merely as a mathematical tool for the exploration of the Equivalence Principle. He justifies this stance by arguing that the significance of the EP lies not in classical physics, but rather in the context of Quantum Theory of Gravitation. [3] He further maintains that it is impossible to construct a quantum theory for a massless spin-two particle that is Lorentz-invariant unless the corresponding classical theory adheres to the EP. Thus, the EP constitutes the most effective bridge between the theory of gravitation and elementary particles, and through a rather unconventional development, he classically derives Einstein's equations. However, his technique has been subject to criticism, for example, by Michael Friedman ([13], p.202).

Leaving controversies aside, it is worth highlighting what Ray D'Inverno ([14], p.131) stated when discussing another of Einstein's principles, the Principle of General Covariance: "although some authors criticize this principle, claiming it is an empty one, [...] this perception only arose after the advent of general relativity." This interesting assessment by D'Inverno regarding the Principle of Covariance may well be extended to Einstein's Equivalence Principle, insofar as the criticisms directed at the EP only became possible after its formulation and the subsequent construction of the General Theory of Relativity. Let us further emphasize that the Principle of Covariance establishes that the form in which physical laws are expressed must be independent of any coordinate system employed.

Over a century since its formulation, relativists, epistemologists, and other researchers continue to write about, discuss, and formulate new versions of the EP. It is also well known that the extensive literature on Einsteinian gravitation generally includes a section dedicated to the principles established by Einstein. In this context, the formulation developed by Friedman [13] has had significant influence on the literature on the subject and was employed by Eduard Prugovečki ([15], pp. 5, 22)

[2]These authors stated, based on their experiment, "...Consequently, we can rule out the existence of repulsive gravity of magnitude 1g between the Earth and anti-matter. The results are thus far in conformity with the predictions of General Relativity. Our results do not support cosmological models relying on repulsive matter–antimatter gravitation."

[3]Weinberg, in ([12], p.537), stated: "This goes a long way toward showing that Einstein's principle of equivalence is a necessary consequence of Lorentz invariance as applied to massless particles of spin two."

in his development of a Quantum Theory of Einsteinian Gravitation. [4] Friedman also pointed out that the strong versions of the EP fail to highlight the distinction between first-order and second-order laws, as we shall examine later in this work. [5] He effectively employed the EP as part of the mathematical structure proposed in his classical (non-quantum) formulation of GTR. [6] This treatment directly involves the EP ([15], pp. 5, 22), and serves as the foundation for his "Quantum General Relativity".

It is important to note that the mathematical tools which contributed to Einstein's final formulation of the theory originated in the nineteenth century, such as tensor calculus. During the nineteenth century, new geometries arose—distinct from that founded by Euclid—which proved fundamental for Einstein's GTR. Here, we shall begin to introduce the author on whom this study is focused: Gottfried Leibniz. In the early eighteenth century, within the context of geometric innovation, we find in the literature texts which make it quite clear that Leibniz, with his "Analysis Situs", made a significant attempt to develop a new geometry, distinct from those previously established by Euclid and Descartes.

According to Echeverría ([17], p.16), Leibniz's view on geometry, in particular his *Analysis Situs*—which he also referred to by other names such as *Characteristica Geometrica* and *Geometria Situs*—was of great relevance to subsequent geometric conceptions. Vincenzo De Risi ([18], p.9) likewise emphasized that "Many scientists and philosophers claimed that they understood what Leibniz had envisaged with his studies; Wolff, Euler, Buffon, Lambert, Kant, Carnot, Grafmann, Klein, Riemann, Poincaré: all of them relied on Leibniz to illustrate their own attempts to construct a new geometry and sometimes a new metaphysics that hinged on mathematical results."

We also find precursor ideas of Leibniz in the insightful book by Harvey R. Brown ([19], p.13), in which he emphasizes that Einstein, in 1915, did something analogous to what Leibniz had done two centuries earlier. Leibniz rejected a cosmological solution based on his two principles: the Principle of the Identity of Indiscernibles (PII) and the Principle of Sufficient Reason (PSR). Brown highlighted that Einstein was "adopting a stance with clear echoes of Leibniz."

More recently further works reinforce this view presented by Harvey Brown. For example, both Spekkens [20] and Smolin [21] assert that the PII may be regarded as a principle of physics. In [20], this issue is treated more on the ontological level, emphasizing Leibniz's PII in the context of Einstein's work.

At a more mathematical level, we find particularly interesting approaches by Mormann ([22], p.6; [23], p.148), who relates the PII to topological axioms of separability. The issue of indiscernibility and its implications for the dichotomy between Newtonian absolutism and Leibnizian relationalism has been discussed by several authors, including Earman [24], Stein [25], Friedman [13], Brown [19], and Norton [16], among others.

We note that Stein ([25], p.379), in his ontological discussion of Leibniz's relationalism, stated that "This is the doctrine that has made Leibniz's view fascinating to me"... and moreover, he critically examined other aspects of Leibniz's thought ([25], p.3).

Throughout our studies on GTR, we observed that new topics could be added to the existing literature. For instance, issues related to the infinitesimal formulation of the EP, more specifically concerning post-Pauli versions of the EP [7], as this version is the most suitable for our purposes.

In investigating the relevance of these themes, we found that a re-examination of the questions addressed by these authors would be appropriate, with the addition of certain points related to the EP. Among these, we shall discuss Leibniz's Indiscernibility and Infinitesimals, which, directly or indirectly, laid the groundwork for the establishment of new geometric concepts within GTR. We

[4] Building upon Friedman's formulation, Prugovečki ([15], p.52) develops the following statement: "In any Lorentz moving frame which is inertial for some timelike geodesic γ, all the non-gravitational laws of physics, expressed in the normal coordinates associated with that inertial frame, should at each point along γ equal, up to first-order terms in those coordinates, their special relativistic counterparts expressed in the tensor coordinates associated with the respective Lorentz frames."

[5] In addition to Friedman's approach, we also find in the same vein the important and enlightening treatment of the EP by John Norton [16].

[6] In [15], he adopted Friedman's definitions of the EP, and in Section 2.6 (p.52) of [15], one also finds elaborations based on this strong formulation of the EP.

[7] W. Pauli, in his book on the Theory of Relativity—first published in 1921—proposed an infinitesimal version of the EP. We shall make use of the English edition from 1958 ([26], p.145).

shall examine, in a concrete manner, how these issues are present in GTR, for instance, in the parallel transport of vectors and in Schild's ladder.

Our work is organized as follows: in Section 2, we address the topic of the Equivalence Principle, which will be explored in greater detail, including a brief historiographical overview as well as an examination of various formulations of the principle. In Section 3, we shall deal with Leibniz's Principle of the Identity of Indiscernibles (PII). Section 4 will discuss Leibniz's infinitesimals in conjunction with the PII. In Section 5, we aim to suggest possible repercussions of Leibnizian concepts and ideas—specifically infinitesimals—on techniques present both in the EP and in topics typical of Einstein's GTR, such as the parallel transport of vectors and Schild's Ladder, a method for parallel transport in a region infinitesimally close to a given point. In the final section, we shall present a brief summary of some conclusions drawn from our approach.

2 On the Equivalence Principle

We shall here provide a description of the Principle of Equivalence, highlighting facts that are relevant to our central objectives, which will be addressed in Section 5. We recall that, following the development of the Special Theory of Relativity (STR) in 1905, Einstein embarked on a path that would lead him to the Principle of Equivalence and to his Theory of Gravitation. As he himself later stated [8]

> After the special theory of relativity had shown the equivalence for formulating the laws of nature of all so-called inertial systems (1905) the question of whether a more general equivalence of coordinate systems existed was an obvious one.

This is what he stated in the famous lecture delivered at the University of Glasgow in 1933 (*First Lecture on the George A. Gibson Foundation in the University of Glasgow Delivered on June 20th, 1933*).

Prior to the STR, various models of the Æther prevailed in Physics. In the case of gravitation, as emphasized by J. Renn and M. Schemmel ([27], p.4), the "aether is imagined to consist of particles that move randomly in all directions. Whenever such an aether atom hits a material body it pushes the body in the direction of its movement".

There are authors such as J. B. Barbour ([27], p.591), for example, who consider that Einstein "banished" the Æther from the foundations of Physics:

> Having banished the aether from the foundations of physics, Einstein felt that he had made an important first step on the way to the complete elimination of the notion of absolute space.

However, it is known that the Æther was later reconsidered in certain contexts, including by Dirac [28], who developed a study re-examining it within the quantum framework. In this view, its velocity is also treated as a quantum observable and, as such, is subject to the uncertainty relations: "[...] will be distributed over various possible values according to a probability law obtained by taking the square of the modulus of a wave function." These investigations gave rise to further approaches to the Æther, including in cosmological settings, as in [29].

The idea of generalizing the Special Theory of Relativity [9] was one of Einstein's objectives after 1905. As discussed in the literature, in 1912 Einstein focused on two fundamental points: covariance—that is, the invariance of the laws of Physics under general coordinate transformations—and the "hypothesis of equivalence", as the Principle of Equivalence (PE) was initially termed ([31], p.151). [10] It should be emphasized that, in later reflections, Einstein himself ([34], p.421) stated that the Principle of Equivalence was the "happiest thought of [his] life".

Despite all this, criticisms of the principle have always existed. One example is the well-known statement by Synge ([35], p.ix), in which he stated,

[8] Excerpt taken from J. Barbour in ([27], p.592).

[9] An important reference regarding these developments is [30].

[10] Since that period, the "hypothesis of equivalence" has already been subject to criticism, for example, by G. Mie in 1914 ([32], pp.699–728; p.727) and L. Silberstein in 1918 ([33], p.94).

> The Principle of Equivalence performed the essential office of midwife at the birth of general relativity, but, as Einstein remarked, the infant would never have got beyond its long-clothes had it not been for Minkowski's concept. I suggest that the midwife be now buried with appropriate honors and the facts of absolute space-time faced.

We may now turn to the formulation of the PE that is most relevant to our objectives: the infinitesimal formulation of Pauli ([26], p.145). However, in ([16], p.203), Einstein's criticisms of this formulation are discussed. It is worth noting that infinitesimal regions of space-time had already been previously mentioned by Einstein himself in formulations of the PE. See, for instance, in 1916 ([36], p.118), shortly after formulating the GTR, he stated: "For infinitely small four-dimensional regions the theory of relativity in the restricted sense is appropriate, if coordinates are suitably chosen."

The restriction, *the theory of relativity in the restricted sense is appropriate*, can be interpreted in the light of the formulations described by Friedman [13] and by Norton [16], which we shall address later on. This formulation is also commonly referred to as the "strong formulation" of the EP ([15], p.5), which was further developed and rewritten by Friedman ([13], p.202). He thereby made a distinction between infinitesimal laws, which are valid at a point, and local laws, which are valid in the neighborhood of the considered point. It is worth recalling that within the "neighborhood" of a point, under the usual topological definitions, there may exist points that are not infinitesimally close, yet still belong to the same neighborhood. The aforementioned authors (Friedman and Prugovecki) elaborate on the [strong] EP in terms of the orders of physical quantities — in this case, the orders refer to the derivatives [11] of the metric tensor, that is, of g_{mn}. This formulation of the EP will be the focus of our present work.

Friedman's formulation is well known in the literature and has become a standard reference on the subject. A similar formulation is also found in ([16], p.239); however, one must pay attention to its terminology, which differs somewhat from that of Friedman. For instance, quantities that Friedman ([13], pp.188–202) regards as second-order are considered third-order by Norton.

Based on these considerations, we shall also assume that *special relativity and general relativity share the same "infinitesimal" structure, but not the same local structure* ([13], pp.188–202). This distinction asserts that first-order structures determine the properties of the tangent space at each point of the manifold, whereas second-order structures determine how these tangent spaces are interrelated — this is where curvature arises. It is well known that this distinction is fundamental, since the "infinitesimal" structure of a tangent bundle is related to the internal characteristics of each tangent space, but not to the topology of its base manifold, which is reflected only in its "local structure" ([15], p.5). Following ([16], p.239), by considering the different orders of the derivatives of the metric tensor $g_{\mu\nu}$, the following is established:

> When special relativity is said to hold in K_0 in an infinitesimal region around p, what is meant is the following. In K_0, at p, structures defined on the manifold, which do not deal with second and higher (coordinate) derivatives of the metric tensor, behave identically to their special relativistic counterparts at any point of a Minkowski spacetime in a Galilean coordinate system.

It is further emphasized that "The two cases differ however when quantities containing $g_{il,mn}$ are considered. Most notably the curvature tensor vanishes only in the case of Minkowski spacetime." Let us recall that the curvature tensor depends on second-order derivatives of the metric tensor, $g_{il,mn}$.

Thus, as stated in ([13], p.185), in SR, the tangent spaces at different points in spacetime are identical, as are their light cones. In GR, however, this may not hold, due to the presence of curvature, which causes the light cones to "tilt," "expand," or "contract" when moving from one tangent space to another.

These issues addressed by the aforementioned authors, concerning structures of different derivative orders of the metric tensor, are fundamental in these new formulations of the EP. This was emphasised by Prugovecki ([15], p.5): "the standard formulations of the [strong] EP obscure these distinctions between first-order laws and higher-order laws." It should also be noted that, in a different manner, Norton ([16], p.240) stated:

[11] We shall adopt the simplified notation $\frac{\partial g_{mn}}{\partial x^k} \equiv g_{mn,k}$, $\frac{\partial g_{mn}}{\partial x^k \partial x^p} \equiv g_{mn,kp}$, etc.

> First, ambiguous restrictions concerning infinitesimal regions will be replaced by restrictions concerning orders of quantities. The assertion that special relativity holds infinitesimally in general relativity, will be taken to mean only that special relativity holds at a point in the spacetime manifold when quantities up to second-order only are considered.

Friedman's approach has been the subject of extensive discussion in the literature and, more recently, in Harvey Brown's book [19], many of the issues raised by Friedman have been revisited. The considerations made by Brown in his section 9.5.1 (p.169), when discussing "The Local Validity of Special Relativity," are similar to those we shall examine later. It concerns the fact that the derivative of $g_{\mu\nu}$ may vanish at a certain point P, even though the curvature (Riemann tensor) does not vanish, because the second derivatives of $g_{\mu\nu}$ do not vanish at that point. These points are closely related to what Prugovecki ([15], p.52) described differently ([37], p.181): "[...] local Lorentz frame, which therefore operationally defines local inertial coordinates x^{μ}."

For completeness, we also highlight that in the traditional book on Gravitation ([38], p.18), it is stated that "It is difficult to cite any easily realizable device that more fully illustrates the meaning of the term 'local Lorentz frame.'" In our context, we may briefly define a Lorentz frame as a local inertial frame, that is, one defined in a very small region of spacetime.

Due to its relation with the EP, there naturally arises the interesting question of *permanent gravitational fields and non-permanent gravitational fields* [12] R. C. Tolman discussed this issue in his book ([39], pp.174–175), emphasising the difference between these two types of fields. This topic was also addressed by C. Moller in his 1952 book on the "Theory of Relativity", where, in discussing the EP ([40], p.221), he referred to permanent fields:

> It is true that the gravitational fields due to the distant masses can be made to disappear by a suitable choice of the system of reference, viz. by choosing a system of inertia as system of reference, while the gravitational fields arising from 'close' masses such as that of the earth or the sun cannot be 'transformed away' by a proper choice of the system of reference; the latter fields will therefore be referred to as permanent gravitational fields.

Moller further added (p.264) "According to the principle of equivalence there should, however, be no essential difference between permanent and non-permanent fields, both types of fields satisfying the same fundamental laws." In the case of non-permanent gravitational fields, he admitted the existence of coordinates in which the metric assumes the Minkowski form (see p.233, equation VIII-40, and his remarks on p.276). Although his approach differs from the one we are considering, he emphasized (p.274), in the context of the EP and when addressing *local systems of inertia*, that:

> In general, it is not possible to introduce a system of coordinates which makes the components of the metric tensor independent of the coordinates, but, as we shall see now, we can always ensure that this is approximately true in the immediate surroundings of a given point P in 4-space.

We may therefore introduce, at a point P, "geodesic" or "inertial" coordinates, and within them obtain a constant metric tensor, that is, the Minkowski metric of Special Relativity. He thus concluded (p.276) that "In accordance with the principle of equivalence is now assumed that all laws of nature at the point P have the same form as in the special theory of relativity when expressed in terms of the local pseudo-Cartesian coordinates [...]."

Before concluding this section, for the interested reader, we shall mention a few additional interesting discussions from the literature regarding the EP. Long after Pauli's version, other versions of the EP emerged as a natural response to concerns about tidal effects, which depend on the properties of gravitational fields. These are the so-called "punctual" versions, which, according to E. Knox ([41], p.351), first appeared in 1977 with H. Ohanian ([42], p.905), who discussed the [strong] version of the EP, addressing important questions related to Quantum Mechanics. One such issue concerns the fact that the Uncertainty Principle imposes limitations on measurements due to the quantized nature of microscopic systems (see details in [43]).

[12] Permanent fields are defined as those near masses that cannot be "transformed away" by an appropriate choice of reference frame ([39], p.175 and [40], p.221).

The punctual versions arise as a natural response to the concern that tidal effects never truly disappear at any specific scale. Returning to Einstein's 1916 statements ([36], p.118), when referring to GR, he employed the expression that its equivalence to SR is "in the restricted sense is appropriate." In this same spirit, in the punctual version of the EP found in ([44], p.43), the following definition is proposed for SR to be valid, in the restricted sense, at a point:

> **Special relativity holds at** p: there exists a local chart x^{μ} of a neighborhood of p such that the fundamental dynamical and curvature-free special relativistic laws hold in their standard vectorial form in x^{μ} at p.

Accordingly, the following is defined:

> **Punctual Equivalence Principle (PEP)**: for all $p \in M$, special relativity holds at p, in the restricted sense given above.

We also encounter the formulation in ([41], p.353) referred to as the "Effective Strong Equivalence Principle," which involves *emergent spacetime structure*, where, as described in [45], the meaning of the EP in such theories is that "acceleration and gravity are both emergent phenomena," that is, gravitation is identified as an entropic force caused by changes in the information associated with the positions of material bodies.

We shall conclude this section by emphasizing that we assume the [strong] version of the EP, incorporating the considerations on the orders of structure previously discussed. We highlight how SR is valid when considered within the framework of GR. Fundamentally, we take into account the physical-mathematical issues developed by Friedman, Prugovecki, and Norton, stating that the [strong] EP, thus formulated, asserts that the two theories are only equivalent at first order in the derivatives of g. We stress that in Friedman's formulation ([13], p.29), in a more epistemological tone, it is stated: "both the principle of special relativity and the equivalence principle involve the identity of indiscernibles." In a similar vein, this was also highlighted by Reichenbach ([46], p.210), who underscored the importance of the PII for the theory of relativity, stating: "Leibniz expressed this idea in his *principle of the identity of indiscernibles*, from which he derived a theory of the relativity of motion, which even today forms the basis of the theory of relativity." We find that these questions could be revisited in a distinct manner, with examples not yet discussed in the literature, which we shall undertake in the following section.

3 Indiscernibility

The greatness of Newton was also the greatness of the controversies surrounding the concepts of absolute versus relative space and time. As the literature reveals—from the famous correspondence between Clarke and Leibniz to more recent discussions—this theme has remained the subject of fascinating debates. We also know that, with the emergence of Einsteinian spacetime, these discussions intensified, and the already vast bibliography expanded even further. Max Jammer [47], for instance, provides a harmonious account of those centuries of controversy among the ideas of thinkers such as Newton, Huygens, Leibniz, and their respective followers.

Merely for the sake of completeness in this discussion, let us recall that Einstein, in the preface to Jammer's book ([47], p. xvii) on *Concepts of Space*, acknowledged the victory of Leibniz's relative space over Newton's absolute space. In the clash between these two schools of thought, the PII was one of the main arguments Leibniz used to oppose, among other notions, the Newtonian conception of absolute space and time, while simultaneously defending his own view that space is something purely relative—namely, an order relation among the bodies that occupy it. As relevant as this theme is for the history of science and epistemology, it is not our intention to revisit the controversies between Leibniz and Newton, nor Einstein's position in the conflict between Newtonian absolutism and Leibnizian relativism. Therefore, we leave to the interested reader the reflection offered by Friedman ([13], p.204):

> Nevertheless, it is well worth our while to get as clear as possible about the reasons that led Einstein to think (and hope) that his theory realized a relationalist conception, for these reasons present a fascinating tangle of physical, mathematical, and philosophical ideas that is perhaps unique in the history of science.

Amidst this diversity of physical, mathematical, and philosophical concepts to which Friedman referred lies the complex relationship between a physical principle—the EP—and a metaphysical principle—the PII. Although the relationship between these two principles is neither simple nor immediate, it is not arbitrary if we consider that Leibniz himself ([48], p.22) had already stated: "5. The great principles of sufficient reason and of the identity of indiscernibles change the state of metaphysics. That science becomes real and demonstrative by means of these principles, whereas before it did generally consist in empty words."

This passage shows us that the PII and the PSR are two principles which, besides complementing one another, enable the transition from superficial knowledge to scientific knowledge of facts. To illustrate this point, let us recall the example from Leibniz ([48], p.22), in which he states that "There is no such thing as two individuals indiscernible from each other. [...] Two drops of water or milk, viewed with a microscope, will appear distinguishable from each other. This is an argument against atoms, which are confuted, as well as a vacuum, by the principles of true metaphysics."

From this example, we understand that indiscernibility is a transitory state that occurs when one has not advanced sufficiently in the analysis of things to the point of revealing their differences. The issue is that it is not always within our reach to perceive the subtle differences that allow us to distinguish the things around us; and when this happens, we enter a state of indiscernibility. However, this does not mean that no differences exist — they merely remain hidden due to the limitations of our senses. In this assertion, the PSR is implicitly present, as Leibniz states ([49], p.151): "[...] that nothing ever comes to pass without there being a cause or at least a reason determining it [...] and although more often than not we are insufficiently acquainted with these determinant reasons, we perceive nevertheless that there are such."

As seen in this example, Leibniz's arguments on indiscernibility are, to a large extent, linked to the principle of reason. This is further confirmed in his fourth correspondence ([48], p.23-24) with Clarke:

> To say that God can cause the whole universe to move forward in a right line or in any other line, without making otherwise any alteration in it, is another chimerical supposition. For two states indiscernible from each other are the same state, and consequently, it is a change without any change. Besides, there is neither rhyme nor reason in it. But God does nothing without reason, and it is impossible that there should be any here. Besides, it would be agendo nihil agere, as I have just now said, because of the indiscernibility.

Here we find an explicit argument by which Leibniz sought to refute the Newtonian conception of an absolute space that is entirely uniform and immobile. However, it was not only the uniformity of space that Leibniz sought to reject, but also Newton's assertion that ([50], p.79): the parts of space cannot be seen, or distinguished from one another by our senses." [13] This is considered to lead to certain difficulties, if we take into account that our senses and, consequently, our perceptions vary from individual to individual. Leibniz [52], in several passages of the "Monadology", helps us to understand that each monad is unique and distinguished from the others by the knowledge and degrees of perception they possess regarding that which is observed. We also have Einstein's statement ([53], p.2): By the aid of language different individuals can, to a certain extent, compare their experiences. Then it turns out that certain sense perceptions of different individuals correspond to each other, while for other sense perceptions no such correspondence can be established."

This statement by Einstein brings us to the following Leibnizian reflection ([52], p.234):

> Et, comme une même ville regardée de différents côtés paraît tout autre, et est comme multipliée perspectivement ; il arrive de mêmequepar la multitude infinie des substances simples, il y a comme autant de différents Univers, qui ne sont pourtant que les perspectives d'un seul selon les différents points de vue de chaque Monade.

[13] According to B. Mates ([51], p.233), this statement seems to contain two interconnected arguments: one showing that the assumption of absolute space would violate a form of the so-called Principle of Sufficient Reason, and the other showing that it would violate the Principle of the Identity of Indiscernibles. We know that Leibniz assumed the Principle of the Identity of Indiscernibles to be a consequence of the Principle of Sufficient Reason.

Thus, indiscernibility arises from the limitations of our perceptions, which may vary from individual to individual.

From a philosophical perspective, we may understand, according to Leibniz ([54], p.120), that ideas dependent upon the senses do not always provide a means of distinguishing what they truly involve. It is like distant objects that appear rounded because we cannot discern their angles, although we may have some vague impression of them. At the opposite extreme, this thought of Leibniz also applies to very small regions, where, in an infinitely small area, a curved line, for example, may be considered as a straight line, or a curved surface as a plane.

In one version of the PII, Leibniz ([55], p.8; our translation) makes it clear that indiscernibility is admissible only in abstract terms, for example, in the concepts of geometry and pure mathematics in general [14]:

> It is not possible for two things to differ from each other just by the place they occupy at a given time, but there is always the need for some other internal difference to intervene. Thus, there cannot be two atoms with a similar shape and equal size, for example, as if they were two equal cubes. Such notions are mathematical, that is, abstract and not real; everything that is different must be distinguished by something, and mere position is not enough to distinguish them.

In short, it is clear from this passage that Leibniz establishes a disconnection between entities of the physical world and those of geometry. The former are always governed by the PII — that is, in nature there are no two or more indiscernible things; this is only possible for the fundamental concepts of geometry, such as congruence and similarity, for instance. However, these concepts are purely abstract and include notions that Leibniz considers incompatible with nature itself, such as the ideas of a fixed, totally homogeneous space and of uniform rectilinear motions. For Leibniz [54], these ideas are not in accordance with natural laws and are a consequence of our lack of attention to what is imperceptible, which causes them to go unnoticed. He further adds that that which remains hidden does not cease to exist. We believe that this thought, albeit succinct, aptly captures the core of the PII.

4 Infinitesimal Approximations

In this section, we aim to establish connections between Leibnizian concepts and those later defined in STR and GTR. We begin by noting that Einstein ([56], p.139), when discussing some consequences of his equivalence hypothesis, establishes connections between STR and GTR. He illustrates this connection with a comparison between the flat Euclidean geometry and the Gaussian geometry of curved surfaces. In this comparison, he states that the properties of these geometries are equivalent and that, in an infinitesimal region of space, a *curved surface approximates a flat surface*.

This is the issue we intend to discuss here from the perspective of Leibnizian infinitesimals ([57], p.544), which, by definition, imply, among other notions, the ideas of approximation, equivalence, and coincidence as an infinitesimally small distance. [15] However, before highlighting these notions underlying Leibniz's differential calculus, let us look in more detail at how infinitesimals appear in the context of Einstein's *equivalence hypothesis* ([56], p.140):

> The theory of special relativity, therefore, applies only to a limiting case that is nowhere precisely realized in the real world. Nevertheless, this limiting case (also) is of fundamental significance for the theory of general relativity; because the fact from which we started out, namely that no gravitational field exists in the immediate vicinity of a free-falling observer, this very fact shows that in the vicinity of every world point the results of the theory of special relativity are valid (in the infinitesimal) for a suitably chosen local coordinate

[14] According to Parmentier ([17], p.33), geometrical notions do not apply to physical space, for this is always guided by the Principle of the Identity of Indiscernibles.

[15] These highlighted notions are based on the principle of continuity; however, we will not discuss this principle here. But let us see what Leibniz says about the law of continuity: *I have been asked several times to confirm the foundations of our calculus by demonstrations, and here I indicated below its fundamental principles, [...]. For I have, besides the infinitesimal mathematical calculus, also a method in Physics, and both I include under the Law of Continuity.*

> system. [...] In its geometrical properties, a small piece of the surface better approximates those of the plane the smaller it is <an infinitesimally small piece of the curved surface approximates in its properties, without limit, those of an infinitesimal piece of the plane>.

We are aware that these connections between the geometrical properties of STR and GTR are justified through physical and geometrical foundations based on infinitesimally small quantities. [16] However, it is necessary to consider that these infinitesimal quantities are not restricted solely to the physical-mathematical perspective, but also to epistemological ones. In this regard, Weyl ([58], p.92) emphasized: "The principle of gaining knowledge of the external world from the behavior of its infinitesimal parts is the mainspring of the theory of knowledge in infinitesimal physics as in Riemann's geometry."

Therefore, we believe it is relevant to highlight, in this brief discussion on infinitesimals, some of their foundations, considering the pertinence of this notion for the equivalence hypothesis and, to some extent, in the relations discussed here between STR and GTR. Jammer ([47], p.158), for example, highlighted the importance of infinitesimals for Riemannian geometry by stating about this geometry: "Being essentially a geometry of infinitely near points, it conforms to the Leibnizian idea of continuity principle." [17] In this passage, Jammer was very precise in stressing the link between Leibnizian infinitesimals and the law of continuity. This is confirmed in the letter that Leibniz sent to Varignon in 1702 [57], seeking to justify the foundations of his differential calculus,

> [...] even if someone refuses to admit infinitesimals, in a strict metaphysical sense as real things, they may still be used safely as ideal concepts [...] Infinitesimals are founded in such a way that everything in geometry and even in nature happens as if they were perfect realities. Proven not only in our geometric analysis of curves but also in my law of continuity, by virtue of which we may consider rest as an infinitely small motion [...], coincidence as an infinitely small distance, equality as the limit of inequalities, etc.

These Leibnizian examples are grounded in his law of continuity [54], which asserts that nothing occurs abruptly or by jumps. Every change or approximation, whether from a greater to a lesser magnitude or vice versa, takes place gradually. In the approximation between a regular polygon and a circumference, for instance, Leibniz ([57], p.546) acknowledged that, although it is not strictly true that a circle is a kind of regular polygon, it may nonetheless be said that the polygon becomes equivalent to the circle through a continuous approximation, wherein the difference between them becomes infinitesimally small and tends to vanish. In this same sense, albeit in another context, Reichenbach ([59], p.142) stated that "a non-Euclidean Riemannian space, considered through infinitesimally small elements, is approximately Euclidean, though not rigorously so." [18]

These infinitesimally small elements to which Reichenbach referred take us back to the foundations of Leibnizian differential calculus. For example, when Leibniz ([60], p.323) states that "the differential or infinitesimal quantities", such as dx or dy, express an infinitely small distance between any two points on a curve. As the differentials are not fixed [57], they allow for ever greater approximations such that the distance between points on the curve may be practically regarded as null, though not entirely vanishing. Thus, dx, d^2x, d^3x, etc., signify the first, second, and third differences and, as a consequence, the greater the order of the differential, the smaller the difference between two points and the greater the approximation between them ([60], pp.377–382).

It is clear that in these comparisons we are dealing with quite distinct realities from the problems addressed by Leibniz, Reichenbach, and Einstein. Nevertheless, the basic idea is implied that infinitesimals are quantities ([52], p.380) which, virtually, express the possibility of approximating two distinct points such that the difference between them becomes so small that it may be neglected.

[16] In the infinitesimal formulation of the EP that we are using ([13], p.212 and [15], p.5), in the equivalence between GTR and STR, we must note that it does not apply to local laws, which are valid in a neighborhood of the considered point. In this local structure, the topology of the manifold is reflected, since "neighborhood" is a topological concept and may not be "infinitesimal".

[17] It is interesting to note that in this statement, Jammer relates the principle of continuity to Leibnizian infinitesimals and Riemannian geometry, which, in turn, was crucial for GTR.

[18] This reasoning is of particular interest, as it is known that in spaces with curvature the Riemann tensor is non-zero, as strongly emphasized by Synge ([35], p.ix), when he affirms: "In Einstein's theory, either there is a gravitational field or there is none, according as the Riemann tensor does not or does vanish. This is an absolute property."

Despite all the difficulties Leibniz [57] faced concerning the foundations of his differential calculus—more specifically regarding the notion of infinitesimals—he consistently sought to reaffirm in his manuscripts that his method was rigorously grounded and substantiated through geometrical demonstrations. For this reason, his infinitesimal calculus was independent of any metaphysical controversy.

5 Reflections in GTR

The influence of Leibniz on developments in Physics has long been noted by various authors. With regard to Relativity, Evangelides ([62], p.1), for instance, expressed this by stating that Leibniz *foresaw Relativity*.

Having now examined in more detail both the PII and Leibniz's infinitesimals, we shall consider possible implications of these concepts through the infinitesimal formulation of the EP as previously discussed. We shall seek to reveal such connections in the techniques of parallel transport and Schild's ladder, emphasizing these aspects in the parallel transport of vectors in curved spaces, as well as in Schild's ladder. To that end, a brief introduction to these topics is required before drawing analogies with Leibniz's formulations.

5.1 Regarding the Principle of Equivalence

Starting from the formulation of the EP, and following the accounts of Friedman ([13], p.202) and Norton ([16], pp.241–242), we write that, to first order, SR and GR share the same "infinitesimal" structure, or in other words, possess the "deceptive appearance" of being identical. Moreover, as noted by Norton ([16], pp.239–240): *It is now clear that the notion of these infinitesimal regions is problematic in differential geometry, since such regions cannot be equated with neighborhoods in their usual sense or any other structure commonly employed.*

Therefore, for the consistency of the formulation, one cannot assume the "infinitesimal region" without specifying the order of the structures considered in the manifold, in this case with respect to the derivatives of the metric tensor $g_{\mu\nu}$. In this way, to first order, the curvature of the manifold is neglected, but if we consider second-order terms, that is, the second derivatives of the metric tensor, curvature must be taken into account [19]. We may say, somewhat loosely, that over infinitesimal intervals, "at first order, curvature is undetectable".

Consequently, considering Leibniz and his PII, and reflecting the general suggestion of Smolin previously cited in [21], we assert within the scope of GR that curvature reveals that two points, even if infinitesimally close, have a '*hidden*' difference. For when second-order terms and thus curvature are considered, this equivalence no longer holds — the equations of the two theories diverge. These considerations allow us to highlight these "reflections" of the PII within the [strong] Equivalence Principle.

5.2 Two Techniques for the Parallel Transport of Vectors

Vector equations in physics involve the differentiation of vectors, which requires computing the difference between vectors at nearby points. However, operations defined on vectors, such as addition, subtraction, and others, are only defined at the same point. It is considered that vectors located at different points belong to different tangent spaces [20].

To compare two vectors at different points, one of them must be carried from one point to the other, so that a comparison may be made—analogously to how it is done in flat spaces—only then can they be meaningfully compared. In curved spaces, this process is known as the *parallel transport* of a vector from one point to another. In this "transport", it is desired that the vector remains "constant in direction and magnitude" ([64], p.231), in analogy with what occurs in "flat" or curvature-free spaces. Consequently, the differentiation of vectors will involve this type of transport, so that both vectors reside in the same tangent space.

[19] At this point, we recall that J. L. Synge's criticisms of the EP, mentioned in our previous sections, were formulated prior to these later versions of the EP.

[20] See ([63], p.430) for an introduction to these topics.

In the discussion of parallel transport within the context of the EP, it is considered ([38], p.207) that:

> *[...] the vehicle between flat spacetime and curved spacetime is the equivalence principle: The laws of physics are the same in any local Lorentz frame of curved spacetime as in a global Lorentz frame of flat space- time. [...] An observer in a local Lorentz frame in curved spacetime can compare vectors and tensors at neighboring events, just as he would in flat spacetime. But to make the comparison, he must parallel-transport them to a common event.*

Our considerations start from the EP assumption in the form detailed in the previous sections. The comparison between vectors and tensors when there is curvature in the studied manifold must be handled with care, as mentioned above. We shall initially define that a local Lorentz system at a given point, or event E_0, is a coordinate system in which the metric $g_{\mu\nu}(E_0) = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric of STR, and thus the first derivatives of $g(E_0)$ vanish. However, the second derivatives (i.e., $g_{\mu\nu,\alpha\beta}(E_0)$) may be non-zero, due to curvature. [21]

In connection with this, as noted in ([38], p.208), it is known that parallel transport can be performed using coordinates; in this case, one must use the Lorentz frame defined above. However, parallel transport can also be performed via Schild's ladder [65], which does not require coordinates or basis vectors ([38], p.208, and Box 10.2, p.248). The latter reference widely disseminated Schild's work, and many authors cite it as a standard reference on Schild's ladder. Others, however, cite ([66], p.63), a work by Ehlers, Pirani, and Schild, which, although it employs Schild's technique, has different objectives. We agree with [67], who, in discussing Schild's method, highlight precisely this issue. This can be verified by a careful reading of ([66], p.63), which, despite containing many illustrations, is not devoted specifically to a detailed description of the well-known "Schild's Ladder". We also note that in [67], not only is Schild's method discussed, but another method—the "pole ladder"—is introduced, presenting other interesting approximations of what they refer to as "ladder methods".

5.2.1 Parallel Transport

Let us consider some key aspects of the parallel transport of vectors. In Figure 1, we illustrate this succinctly: the left-hand side depicts the case of a space without curvature, with points **P** and **Q** at the ends, where a certain vector $\vec{A}$ is parallel transported from point **P** to point **Q** along two different paths, α and β. Owing to the absence of curvature, at point **Q**, both vectors will coincide, that is, $\tilde{\mathbf{A}}_\alpha \equiv \tilde{\mathbf{A}}_\beta$, where the index denotes the path along which the vector was parallel transported. The case with curvature is shown on the right-hand side of the figure. Now the vector $\tilde{\mathbf{A}}$ at point $\mathbf{P}_1$ is parallel transported to point $\mathbf{P}_2$ along path α or path β. The vectors transported along these different paths, subject to different curvatures, will arrive as vectors $\tilde{\mathbf{A}}_\alpha$ or $\tilde{\mathbf{A}}_\beta$, depending on the curvatures along each trajectory, and thus at point $\mathbf{P}_2$ the resulting vectors will no longer coincide: $\tilde{\mathbf{A}}_\alpha \neq \tilde{\mathbf{A}}_\beta$. Before initiating comparisons with "Leibnizian" concepts, we recall some characteristic facts from Einsteinian gravitation. In an early work on Einstein's theory, Richard Tolman and others [68] discussed approximations of GR in the regime of weak gravitational fields, particularly the gravitational field created by light. They demonstrated that even light curves spacetime. Therefore, within the framework of GR, flat space can be seen as an approximation, since a mere *pulse* of light would modify spacetime.

Interpreting this in the Leibnizian context of the PII, we may say that in curved space under Einsteinian gravitation, the two vectors $\mathbf{A}_\alpha$ and $\mathbf{A}_\beta$, which in flat space would be *indistinguishable* after parallel transport, no longer remain so due to curvature. Thus, even an *infinitesimal* parallel displacement reveals their difference. Recalling the work of [68], which rigorously demonstrates via General Relativity that even a single light ray disturbs spacetime, we see that Minkowski space—with zero pseudo-Riemannian curvature—can only be considered an approximation. The curvature of the manifold causes the two objects, previously regarded as indistinguishable, to become discernibly different: *there are no two identical things in nature*.

[21] See *Gravitation* ([38], p.207) and ([63], p.140).

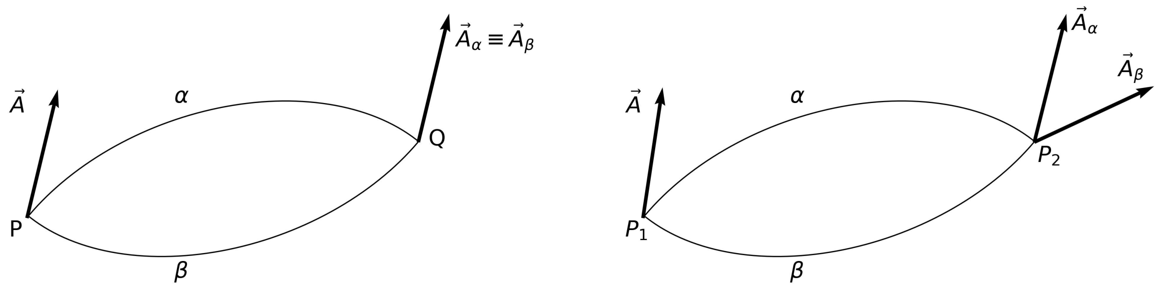

Figure 1: Parallel transport of vectors.

5.2.2 Schild's Ladder and the PII

We shall here summarize the methodology, using the equations most relevant to our conclusions. The full details can be found in the cited references. As mentioned earlier, a method used for the parallel transport of vectors in curved manifolds is the so-called "Schild's Ladder", as it became known in the literature. This was introduced by A. Schild in 1970 at Princeton [65], and became widely recognized through the textbook *Gravitation* ([38], p.248). More recently, Kheyfets et al. [69] presented derivations related to the method, showing that it accounts only for the symmetric part of the manifold's connection.

We emphasize that in [67], it is asserted that the treatment in [69] is limited to first order, and they also provide a rather interesting historiographical account of methods developed after A. Schild's, such as the "pole ladder". This latter method allows for the construction of a third-order expansion of Schild's ladder, with coefficients depending on the curvature tensor of the manifold.

The Schild method was also described in [70] (see Section 3.1), in considerable detail using geodesic parallelograms. Those interested in higher-order approximations will also find them in [67], where applications of the method to medical fields are cited—for instance, in the analysis of tomography data from Alzheimer's patients. Let us recall that all such developments originated from the methodology established by Schild. Here we shall focus on the developments presented in ([38], p.248) and ([69], p.2892), but we shall begin with a summary of the method [22].

Based on the treatment presented in ([69], p.2895), we produced Figure 2, where we consider a manifold **M**, and a geodesic $C(\beta)$, where β is the parameter along the curve, shown in the left-hand diagram. At point P_0, we have the vectors $\vec{v}$ and $\tilde{\mathbf{A}}$, tangent to their respective curves; where $\vec{v}$ points along the curve from P_0 to point **E**, and may be written as $\vec{v} \equiv \partial C(\beta)/\partial\beta$ while $\tilde{\mathbf{A}}$ points from P_0 to point **D**, and is tangent to this curve with parameter α. Now, considering the right-hand diagram, we construct from point P_0, in an infinitesimal region, a parallelogram with its diagonals assigned there [23]. From point P_0 to point P_1, in the direction of vector $\tilde{\mathbf{A}}$, we associate the parameter α. In the direction of P_0P_2, we have vector $\vec{v}$ with parameter β, and now the vector $\vec{k}$ from P_0 to P_3, with parameter λ. We wish to perform the parallel transport of vector $\tilde{\mathbf{A}}$ from point P_0 to point P_2, where the translated vector, after parallel transport, will be denoted $\tilde{\mathbf{A}}_{\|}$, i.e., at point P_2.

In the approximation used in the Schild method, the geodesic in the direction P_0P_3, with parameter λ and local coordinates $\{x^\mu\}$, is written as

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\kappa^\alpha\frac{dx^\beta}{d\lambda} = 0\ , \quad \text{or} \quad \frac{d\vec{\kappa}^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}\kappa^\alpha\frac{dx^\beta}{d\lambda} = 0\ \ , \tag{1}$$

[22] Those interested in further details may consult the cited references.

[23] Following the rules established in ([69], p.2892), "[...] the whole Schild's ladder construction to be placed within one coordinate neighborhood. All coordinate expressions will be written in the chart of this coordinate neighborhood,[...]"

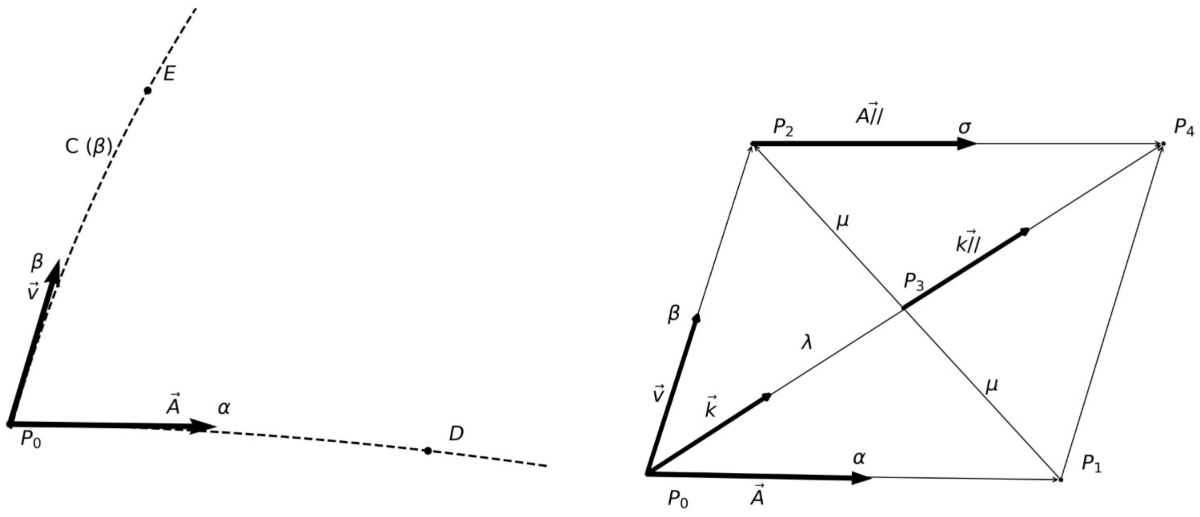

Figure 2: Schild's Ladder.

where $\vec{\kappa}$ is the tangent vector to this geodesic, which may be parallel transported along the geodesic from P_0 to P_3, represented by $\kappa^\alpha \to \kappa^\alpha_{//}$. In the lowest-order approximation, the right-hand side of (1) may be written as

$$\kappa^\alpha_{//} - \kappa^\alpha = -(\Gamma^\mu_{\alpha\beta})_0\, \kappa^\alpha (x^\beta_3 - x^\beta_0)\ , \tag{2}$$

where the indices 0 and 3 on the coordinates x^β refer to points P_0 and P_3, and the subscript 0 on the connection coefficient $(\Gamma^\mu_{\alpha\beta})_0$ indicates it is evaluated at point P_0, where the original vector is located. This index may be suppressed following equation (9) of ([69], p.2896) [24].

Analogously to what was done for the vector κ^α, we shall now proceed for the vector $\mathbf{A}^\mu$, outlining the successive steps of the procedure:

1. The curve from P_0 to P_1 has parameter α, hence we consider the the vector $\frac{\partial}{\partial\alpha}|_{P_0} = \tilde{\mathbf{A}}$, where the infinitesimal separation P_0P_1 is equal to α;

2. From P_0 to P_2, we have the infinitesimal portion of the curve $C(\beta)$, with point P_2 connected to point P_1 via the diagonal of the parallelogram. This diagonal is parametrized by μ, and the segment P_1P_2 will have total value 2μ, with μ starting at P_1 when $\mu = 0$. Halfway along this geodesic, at point P_3, the value of the parameter is μ, and at P_2 it reaches $+2\mu$;

3. Connecting point P_0 to point P_3, we consider the other diagonal, with parameter λ, which at P_3 equals λ and when extended to point P_4 reaches $+2\lambda$. Thus, the parallelogram $P_0P_1P_4P_2$ is constructed, with its diagonals P_0P_4 and P_1P_2 intersecting at P_3;

4. The tangent vector to the curve P_2P_4 will be the the vector $\tilde{\mathbf{A}}_{||}$. i.e., the vector $\tilde{\mathbf{A}}$ (originally at P_0) parallel transported along the geodesic to point P_2;

5. Analogously to what was written in (2), and following the low-order approximations carried out in ([69], pp.2896–2897), a relation is obtained which we shall present below in (3).

From the low-order description of geodesics given in [69], we can conclude that $\mathbf{A}_{||} - \mathbf{A}$ is proportional to the symmetric part of the connection $\Gamma^\alpha_{\mu\nu}$, to the components A^μ of $\mathbf{A}$ and v^β of $\vec{v}$ (see formula (26) in [69]); in our case, instead of u^β from $\vec{u}$, we have v^β from $\vec{v}$, along with the parameter τ, which in our case is β. It thus follows that the method determines a parallel transport with respect to the symmetric

[24] Further details of these approximations may be found in [70], Section 3.1.

part of the connection. For the purposes of our analysis, this does not affect our conclusions, since in the gravitational theory we adopt—Einstein's theory—it is assumed that the connection is symmetric, i.e., $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$. Consequently, we may write:

$$\mathbf{A}^{\mu}_{\|} - \mathbf{A}^{\mu} = -\beta\,\Gamma^{\mu}_{\alpha\beta}\,A^{\alpha}\,v^{\beta} \quad . \tag{3}$$

Let us now turn to our conclusion revisiting Leibniz and his PII. According to GR, we know that in (3), the $\Gamma^{\mu}_{\alpha\beta}$ are *not tensors*, and are in fact first-order derivatives of the metric tensor. Even in manifolds with non-zero curvature, we can always choose at a point P_0 a special coordinate system, $\{(x^{\mu})_0\}$, where the coordinates at that point are commonly referred to as "Galilean coordinates," for which the metric tensor is constant, and the Christoffel symbols $(\Gamma_{\alpha\beta})_0$ vanish. This special coordinate system is also known as a "locally inertial frame" [25]. The apparent flatness of the manifold in the system where the Christoffel symbols vanish is therefore *misleading*, because in another coordinate system the curvature is revealed. Thus, "Leibnizianly" speaking, it is analogous to two drops of water which, although apparently identical, are discernible in their finer details.

We consider a *local Lorentz frame* at a given event E_0 as a reference coordinate system in which the metric $g_{\mu\nu}(E_0) = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric of the STR, and therefore the first derivatives of $g_{\mu\nu}(E_0)$ are zero. Furthermore, the second derivatives (or $g_{\mu\nu,\alpha\beta}(E_0)$) may not vanish, due to the curvature ([38], p.207).

We may then assert that even in curved space, using Schild's method as summarized in (3), by choosing at point P_0 a locally inertial coordinate system, i.e., $(\Gamma^{\mu}_{\alpha\beta})_0 \equiv 0$, we have $\mathbf{A}^{\mu}_{\|} = \mathbf{A}^{\mu}$. However, if we choose any other "non-inertial" coordinate system the *deceptive appearance* of flatness is revealed, and $\mathbf{A}^{\mu}_{\|} - \mathbf{A}^{\mu} \neq 0$. We may therefore say this is in agreement with Leibniz's PII, because the equality of $\mathbf{A}^{\mu}_{\|}$ with $\mathbf{A}^{\mu}$ does not in fact hold [26].

6 Conclusion

The central objective of this work was to discuss the Principle of the Identity of Indiscernibles (PII) and Leibniz's formulation of Infinitesimals within the context of Einstein's General Theory of Relativity (GTR). We have shown in the preceding sections how Leibniz's formulations served, directly or indirectly, in the development of Physics, particularly in GTR. This discussion was framed through the infinitesimal formulation of the [strong] Equivalence Principle (EP), suited to our objectives, which considers the distinction between first-order and second-order formulations of physical laws. We addressed the techniques of parallel transport of vectors, particularly through the method of "Schild's ladder." Figures (1) and (2) provide a clearer visualization of the treatment developed. Within the context of the Leibnizian investigations cited, we highlighted the importance in Physics of the Principle of the Identity of Indiscernibles, as well as the ongoing discussion in the literature regarding the "survival" or not of PII in Quantum Mechanics, as for instance in ([72], p.117; [73]), and references therein.

Our development may be synthesized as reflections of PII and of Leibniz's constructions in Einstein's theories. It is natural that, in the evolution of scientific reasoning, certain ideas become consolidated in the minds of new researchers in such a way that their original foundations are no longer always recognized—especially when not directly evident. Our intention was to make these relations explicit, particularly in the comparison between the Special Theory of Relativity and the General Theory of Relativity: it is said that, to first order, these two theories of Einstein have the same "infinitesimal structure," that is, they bear the "deceptive appearance" of being identical. It can thus be asserted that, at first order, they exhibit indiscernible manifestations, whereas at higher orders, they are no longer indiscernible. Therefore, in the Leibnizian verification pursued here, it is essential to determine the appropriate order of the physical quantities involved.

By considering structures beyond the first order, one can explicitly reveal, for instance, the pseudo-Riemannian curvature of the manifold.

[25] See ([63], pp. 140 and 179), or ([71], p.241).

[26] In our discussions involving physical theories and PII, it is enriching to see Spekkens' approach in [20], discussing the ontology and verifiability of physical theories in the context of Leibniz's PII.

Finally, we may say that, alongside the greatness of Newton, we may also recognize the greatness of Leibniz, with his principles and precursor concepts. In the future, we may investigate the relevance of these foundations in other areas of Science.

Acknowledgments

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