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Finding the Cut Locus of Reflected Rays by the Wavefronts

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Abstract - In this paper, we will study the relation between the caustics produced by the wavefronts of rays reflected in plane curves and the cut locus, as the self-intersection points of the wavefronts will generate the cut locus of the caustic. This is related to the study of caustics in general and their cusps, as other authors have made in [1, 2, 3]. Firstly, we will define the wavefronts and other related elements to analyze this statement, then we will use two different methods for evaluating the self-intersection points of the wavefronts, which are useful for determining self-intersections of curves in general. In the end, we will use the results obtained and see their relationship with the confocal conics when the reflection curve is an ellipse.

Keywords - Caustic; Envelope; Plane curves; Wavefront; Cut locus; Self-intersection points; Ellipse; Confocal conics.

1 Introduction

1.1 Setting up the problem

Imagine that for a given curve there is a point which is a light source (called a radiant point) and whose light leaves the point in the form of an incident ray, until this ray bumps into the curve, and eventually the curve reflects the light in the form of a reflected ray. Then, if this is done for all the possible directions for which the light leaves the radiant (i.e. reaching all the points of the curve), the result will be an envelope of reflected rays, also known as focal set or caustic (see Figure 1a). Indeed, using the reflections of lines in a curve is a way of defining a caustic, though we will define it in a different way later on.

It has been conjectured that the caustic by reflection in an ellipse [1], has exactly 4 points. The paper ”Cusps of caustics by reflection in ellipses” [1] takes the approach of evaluating the envelope of reflected rays, whereas in this paper the approach will be to take into account the cut locus. As mentioned above, the cut locus is formed by the self-intersection points of the wavefronts. Then, we will see the relationship between the cut locus and the caustic. Throughout this paper, we will establish the origin as the radiant point. Some of the concepts that we will discuss here throughout the paper are defined in [1]. We will consider mostly ellipses, but several results are applicable for curves in general.

1.2 Incident and Reflected Rays

Firstly, let us define any ellipse in both parametric and implicit form:

$$\boldsymbol{\alpha}(\theta) = (x_0 + a\cos\theta, y_0 + b\sin\theta), \qquad \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \tag{1}$$

During this paper, there are some specific ellipses that we will use for demonstration. Those are:

$$\begin{cases} \boldsymbol{\alpha_1}(\theta) = (1.1\cos\theta + 0.1, \sin\theta + 0.1) \\ \boldsymbol{\alpha_2}(\theta) = (1.1\cos\theta, \sin\theta) \\ \boldsymbol{\alpha_3}(\theta) = (3\cos\theta + 0.1, 2.65\sin\theta + 0.2) \\ \boldsymbol{\alpha_4}(\theta) = (3\cos\theta + 0.8, 2.65\sin\theta + 0.8) \end{cases} \tag{2}$$

Furthermore, the $\boldsymbol{T}$ and $\boldsymbol{N}$ vectors represent the unit tangent vector of the curve and the unit normal vector of the curve, respectively, and can be obtained by

$$\boldsymbol{T}(\theta) = \frac{\boldsymbol{\alpha}'(\theta)}{\|\boldsymbol{\alpha}'(\theta)\|} = (u, v)\ ,\ \boldsymbol{N}(\theta) = (-v, u) \tag{3}$$

In addition, the rays leaving the radiant (light source) towards the curve are the incident rays ($\boldsymbol{r}$), whereas those reflected by the curve are the reflected rays ($\boldsymbol{R}$), and using (3) they can be described as:

$$\boldsymbol{r}(\theta) = \xi\boldsymbol{T} + \eta\boldsymbol{N} = (x_0, y_0) + s(a\cos\theta - x_0, b\sin\theta - y_0), \qquad \boldsymbol{R}(\theta) = \xi\boldsymbol{T} - \eta\boldsymbol{N} \tag{4}$$

where (x_0, y_0) is the radiant point and s is a free parameter (which we will define later). The values of ξ and η can be found with $\xi = \boldsymbol{r}\cdot\boldsymbol{T}$ and $\eta = \boldsymbol{r}\cdot\boldsymbol{N}$, respectively.

1.3 The caustic, the wavefronts and their self-intersections

A possible parameterization for the caustic of a curve (not necessarily an ellipse) is described in [3] given by $\boldsymbol{C}(\theta) = \boldsymbol{\alpha}(\theta) + \rho\boldsymbol{R}(\theta)$, where

$$\rho = \frac{-(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{N}})}{2\kappa\|\boldsymbol{\alpha}\|^2 + \boldsymbol{\alpha}\cdot\hat{\boldsymbol{N}}}$$

and where κ represents the curvature. For any curve $\gamma = (x, y)$, its curvature [4] is given by

$$\kappa = \frac{|x'y'' - x''y'|}{(\sqrt{(x')^2 + (y')^2})^3}$$

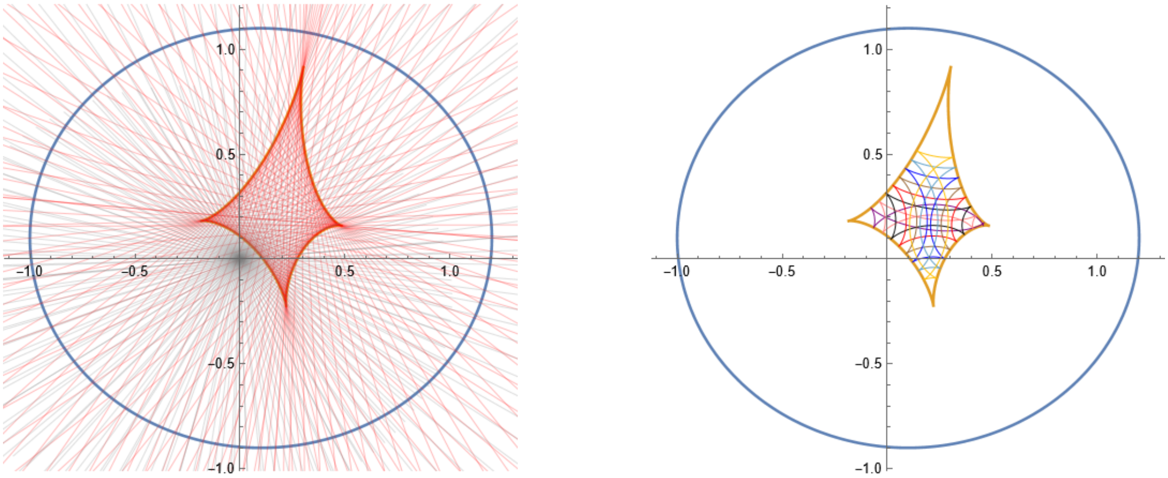
Now, let us define the wavefronts function, using (4). This is defined by a piecewise function, considering whether the parameter $s = T - D(\theta)$ is positive or negative, where $D(\theta)$ represents the length of the incident ray and T represents the total length of both an incident ray and its corresponding reflected ray. If the unit speed is applied, then T represents the time (in seconds) since leaving the radiant point as well. Another thing to consider is that, if an incident ray does not reach the curve, the wavefront will depend on the incident ray, whereas if the incident ray reaches the curve and reflects, then the wavefront will depend on the reflected ray.

$$\boldsymbol{W}_T(\theta) = \begin{cases} \boldsymbol{\alpha}(\theta) + (T - D(\theta))\,\hat{\boldsymbol{R}}(\theta) & \text{if } T > D(\theta) \\ \boldsymbol{\alpha}(\theta) + (T - D(\theta))\,\hat{\boldsymbol{r}}(\theta) & \text{if } T < D(\theta) \end{cases} \tag{5}$$

The wavefronts (5) are important since they change continuously depending on the value of the total length, therefore, for some values of T, they stop being simple curves by showing self-intersections, which are of great interest. In fact, these self-intersections show up when the wavefronts are very close to one of the cusps, as can be appreciated in Figure (1b).

Looking at Figure 1, it can be seen both the ellipse and its corresponding caustic. In the left plot, the caustic is formed by the reflection of numerous incident rays leaving the radiant point, whereas in the right plot, the caustic is formed by the cusps of the wavefronts. In particular, (looking at the right plot) there is a range of values of T such that the wavefronts produced have cusps (and these altogether create the caustic). Besides, for this specific range, some wavefronts will have self-intersections, and others will not.

At a particular value of T, the wavefronts start to self-intersect. If the curve is an ellipse that has its center in the origin, then the self-intersection points start to appear in two of the cusps of the caustic (and one is the opposite of the other one), and then they move to the center (for some values of T) until disappearing for another value of T. Then, for a few more values of T, there are no self-intersection points until another specific value of T is reached, for which the self-intersection points start to show up in the remaining two cusps, moving to the center (for some values of T) until they disappear, ultimately. In fact, the wavefronts and the self-intersection points will be symmetrical in this case.



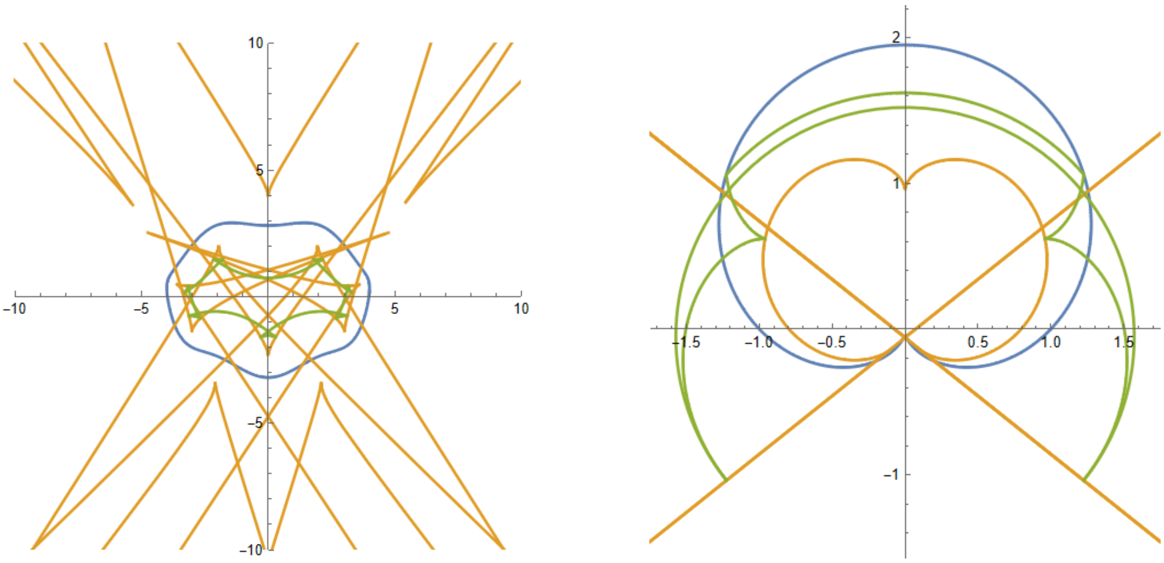
(a) The ellipse $\alpha_1(\theta)$ with incident and reflected rays.

(b) The ellipse $\alpha_1(\theta)$ with its caustic and a set of wavefronts

Figure 1: Comparison between the 2 ways of defining the problem, using the ellipse $\alpha_1(\theta)$ (2)

If the curve is an ellipse with its center displaced from the origin, then there is no symmetry, hence, for a particular value of T the wavefronts start to self-intersect where one of the cusps of the caustic is; then for another value of T the self-intersection of the wavefronts also happens in the opposite cusp, and the self-intersection points move towards the center of the caustic. Then, for some values of T there are no self-intersection points, and for another value of T the wavefronts start to self-intersect again at one of the two remaining cusps, for another value of T it also starts self-intersect at the last cusp. Finally, for some more values of T, the self-intersection points move to the center until eventually disappearing.

More generally, if the curve is not an ellipse, then the caustic and the wavefronts actually can look like something very complicated, though it depends on the curve (see Figure 2).



(a) $\beta_1 = (4\cos\theta + 0.1\sin 2\theta, 3\sin\theta + 0.2\cos 6\theta)$

(b) $\beta_2 = ((1 + 0.95\sin\theta)\cos\theta, (1 + 0.95\sin\theta)\sin\theta)$

Figure 2: Two curves (blue) which are not ellipses, with their caustic (golden) and a specific wavefront (green). There will not be a further exhaustive analysis in this paper for them.

There are two intervals of the values of the total length T that make the wavefronts self-intersect. The first range generates the first set of self-intersection points, whereas the second one gives the second set of self-intersection points. Note that in the following sections of the paper we will only consider the so-called first set of self-intersections because that is the set which defines the cut locus, and thus the second set will not be taken into account.

The next step is to find the self-intersection points, for which there are different methods that could be handy. In fact, the following techniques are valid for detecting self-intersections of plane curves other than wavefronts, which makes this an interesting process to follow, however, wavefronts complicate matters.

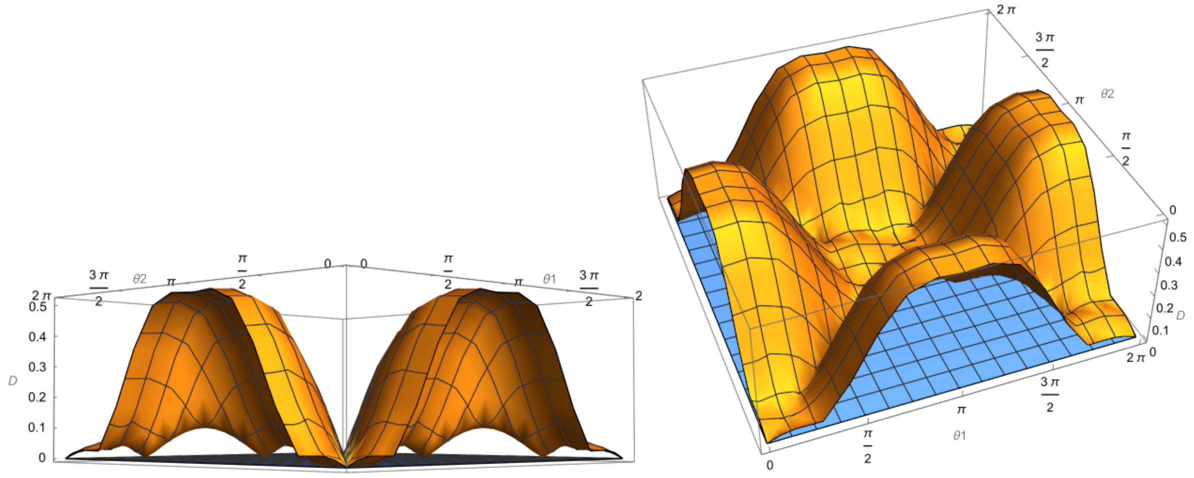
2 The methods for finding self-intersection points

Note: In Section 2, the terms (x_i, y_i) (with $i = 1, 2, 3, 4$) for the first method are different from those for the second method.

2.1 First method

This method might be called "Minimizing the distance between 2 points of the wavefront". Considering that a given curve $\boldsymbol{\alpha}(\theta)$ is parameterized in terms of θ such that $\theta \in (0, 2\pi)$, a self-intersection point satisfies that, for two distinct values of θ, let us say θ_1 and θ_2 such that $\theta_1 \neq \theta_2$, the following expression holds: $\boldsymbol{\alpha}(\theta_1) = \boldsymbol{\alpha}(\theta_2)$.

One of the methods is based on, for this curve $\boldsymbol{\alpha}(\theta)$, selecting a point p (given by θ_1) of the curve, and then for this point p, we select all the other possible points q (given by θ_2). Hence, we can measure the distance between p and q, $D_{pq} = \|p - q\|$, such that $D_{pq} : (p, q) \mapsto \mathbb{R}$. If $D_{pq} = 0$ and the respective values of θ for p and q are far enough away from each other, then this indicates that there is a point of self-intersection, which can be obtained using any of the two possible values of θ.



(a) D_{pq} seen from the front **(b)** D_{pq} seen from above

Figure 3: The $D_{pq} = 0$ plane (blue) and 3D plots of D_{pq} (orange) with different views, using $T = 1.95$ for $\boldsymbol{\alpha_2}(\theta)$.

See Figure 3. Note that the diagonal $\theta_1 = \theta_2$ reaches down to the horizontal plane $D_{pq} = 0$, which makes sense since then p and q are given by the same value, although it does not represent any point of self-intersection. Indeed, looking at the left 3D plot, there are 4 points reaching down the $D_{pq} = 0$ plane.

If we plot the contour of D_{pq} by having θ_1 as its horizontal axis and θ_2 as its vertical axis, such that the value of D_{pq} approaches 0 very close (see Figure 4), then we will appreciate that in the diagonal $\theta_1 = \theta_2$, the contour tends to 0 (which is expected as the distance depends on p and q, and both are given by the same value of θ, however, note that it does not represent any self-intersection points). In addition, it can be seen that near a cusp of the wavefront, the curve folds over on itself and so the points in that neighborhood appear close together but are not self-intersections. Furthermore, the aforementioned diagonal divides the contour into 2 triangles (the upper one and the lower one).

We will be able to see in the contour of $D_{pq} = 0.005$ that, for certain values of the total length T (in particular, for those such that the wavefronts self-intersect), there exist 4 points far away from each other and from the line $\theta_1 = \theta_2$, but with D_{pq} approaching 0 indicates that p and q approach each other. These 4 points are distributed in 2 triangles (2 in the upper triangle and the other 2 in the lower one), such that there exists a pairwise symmetry, with the diagonal formerly mentioned being the axis of symmetry (see Figure 4).

Therefore, both one of the points in the upper triangle, let us say (x_1, y_1), and its symmetric counterpart in the lower triangle, $(x_3, y_3) = (y_1, x_1)$, give one of the self-intersection points (the θ_1 and θ_2 for one of the points are reversed for its symmetric point). The other self-intersection point is given by both the remaining point in the upper triangle, let us say (x_2, y_2), and its symmetric related point in the lower triangle, $(x_4, y_4) = (y_2, x_2)$ (and again, the θ_1 and θ_2 values for one of these are reversed for its symmetric point). Hence, if we find the two

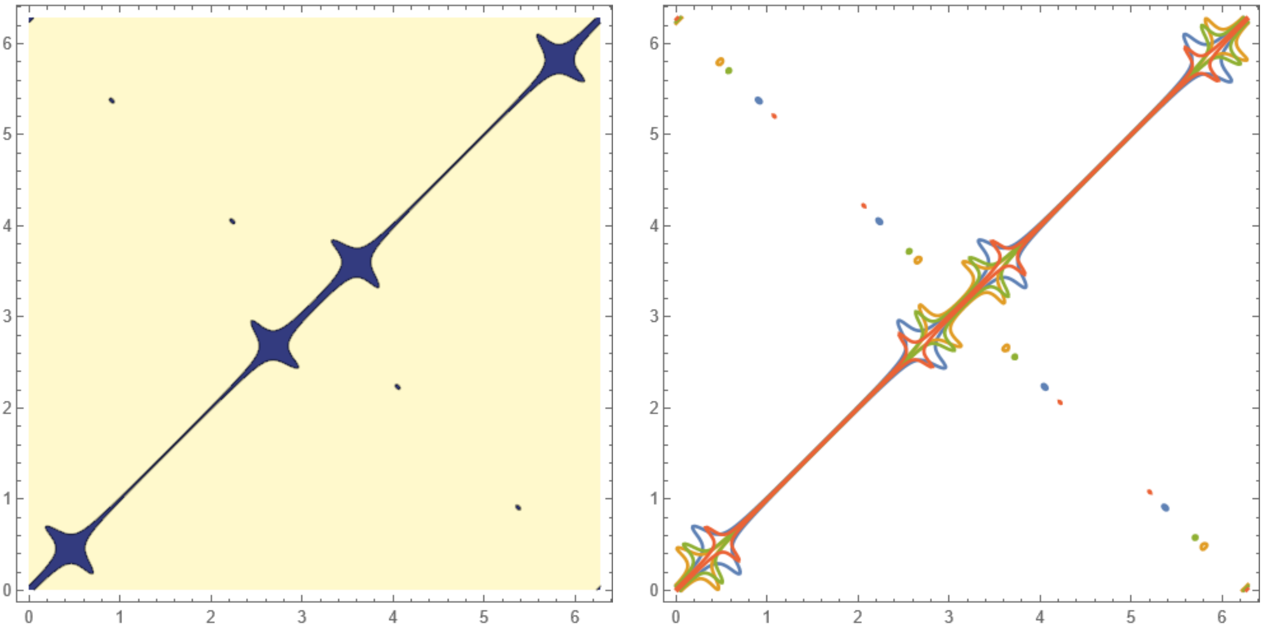


Figure 4: Contour of $D_{pq} = 0.005$ for one (left) and several (right) wavefronts for $\boldsymbol{\alpha_2}(\theta)$. As T increases for the first set of self-intersections, it goes from the yellow one to the red one.

points in the upper triangle, we will have found the values of θ_1 and θ_2 of the 2 self-intersection points, and we will not need to look for the other two in the lower triangle (although it would be easy to find them once we have found the two in the upper triangle); and vice versa.

The Mathematica software has been used to carry out the process, specifically, the core of the code has used the ”Minimize” command (to find the lowest minima) and some ”If” commands, too. The strategy was the following: if we choose the upper triangle (we could have chosen the lower one instead) and exclude all those values close to the equality $\theta_1 = \theta_2$, then we apply the ”Minimize” command to find one of the 2 minima points. Once one of them is found, we exclude both the values close to $\theta_1 = \theta_2$ and also the values inside the disc $(x - x_1)^2 + (y - y_1)^2 \leq 0.01$, and we search again. In this case, we are excluding the search for the already found point, so the new desired value must be somewhere else.

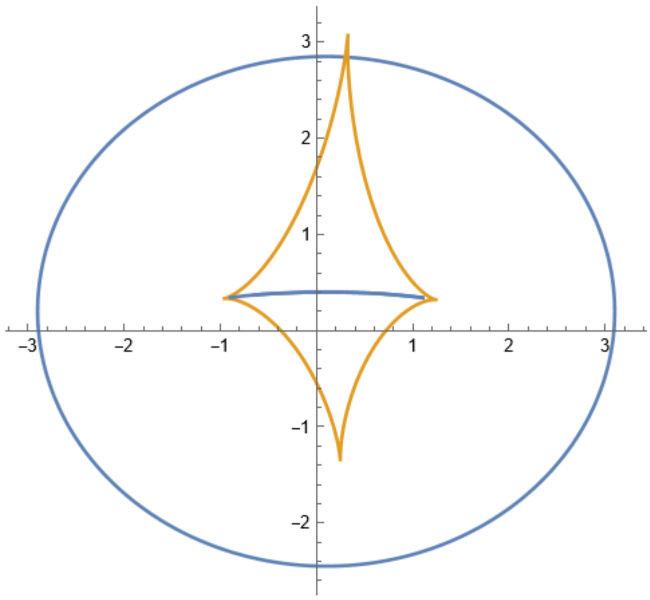


Figure 5: The ellipse $\alpha_3(\theta)$ with its caustic and its cut locus.

Once these 2 points have been found for a particular value of T, we can extract information from them to find the values of θ_1 and θ_2. After repeating this process for numerous of values for T, we can plot the wavefronts function such that it only considers those found values for its respective value, and we will be able to plot the cut locus.

Looking at Figure 5, it can be seen that the self-intersection points of the first set of the wavefronts altogether give the cut locus, a line going from one of the cusps of the caustic to its opposite cusp. This is exactly what we wanted to analyze.

The main advantage of this method is that the code gives the right points and it does not give other minima which are not the global minima of the surface; nevertheless, the main disadvantage is that it really takes a long time to obtain the results. Another advantage is that you can visualize in a 3D plot (like some of the above in Figure 3) how the surface looks like and compare the points obtained with that plot, though another disadvantage is that, when θ_1 and θ_2 are the same value (or very close to each other), in the plot and in the contour, it looks like there might be self-intersection points in the $\theta_1 = \theta_2$ diagonal, though there are none.

2.2 Second method

A distinct approach would be the so-called "Infinite Lines Intersection", which consists of dividing the whole wavefront curve in numerous of segments; then, we check whether their respective infinite lines eventually intersect or they are parallel, and then we analyze whether those intersections occur in the defined segments or not.

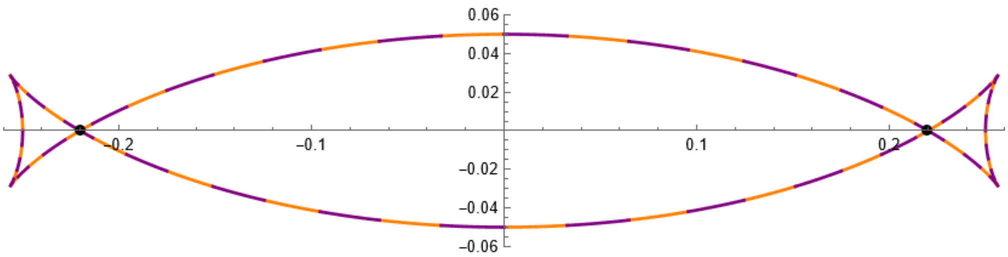


Figure 6: Wavefront of $\boldsymbol{\alpha}_2(\theta)$ with $T = 1.95$. The 2 self-intersection points can be appreciated in black dots.

Let us define two segments such that segment 1 goes from (x_1,y_1) to (x_2,y_2) and segment 2 goes from (x_3,y_3) to (x_4,y_4):

$$\begin{cases} \boldsymbol{r}_1 = (x_1 + w_1(x_2 - x_1), y_1 + w_1(y_2 - y_1)) \\ \boldsymbol{r}_2 = (x_3 + w_2(x_4 - x_3), y_3 + w_2(y_4 - y_3)) \end{cases} \tag{6}$$

for which $w_1, w_2 \in [0,1]^2$. If $w_1, w_2 \in \mathbb{R}^2$, then these two expressions would be infinite lines.

The next step is to see whether the respective infinite lines of the segments in (6) are parallel lines. This will be the case if $(x_2 - x_1) = (x_4 - x_3)$ and $(y_2 - y_1) = (y_4 - y_3)$ hold simultaneously. If the lines are not parallel, then they must intersect at one point of the plane. In that case, we solve for w_1 and w_2 the following equation:

$$\boldsymbol{r}_1 = \boldsymbol{r}_2 \quad \Rightarrow \begin{cases} x_1 + w_1(x_2 - x_1) = x_3 + w_2(x_4 - x_3) \\ y_1 + w_1(y_2 - y_1) = y_3 + w_2(y_4 - y_3) \end{cases} \quad \Rightarrow \begin{cases} w_1 = (\frac{x_3 - x_1}{x_2 - x_1}) + w_2(\frac{x_4 - x_3}{x_2 - x_1}) \\ w_1 = (\frac{y_3 - y_1}{y_2 - y_1}) + w_2(\frac{y_4 - y_3}{y_2 - y_1}) \end{cases}$$

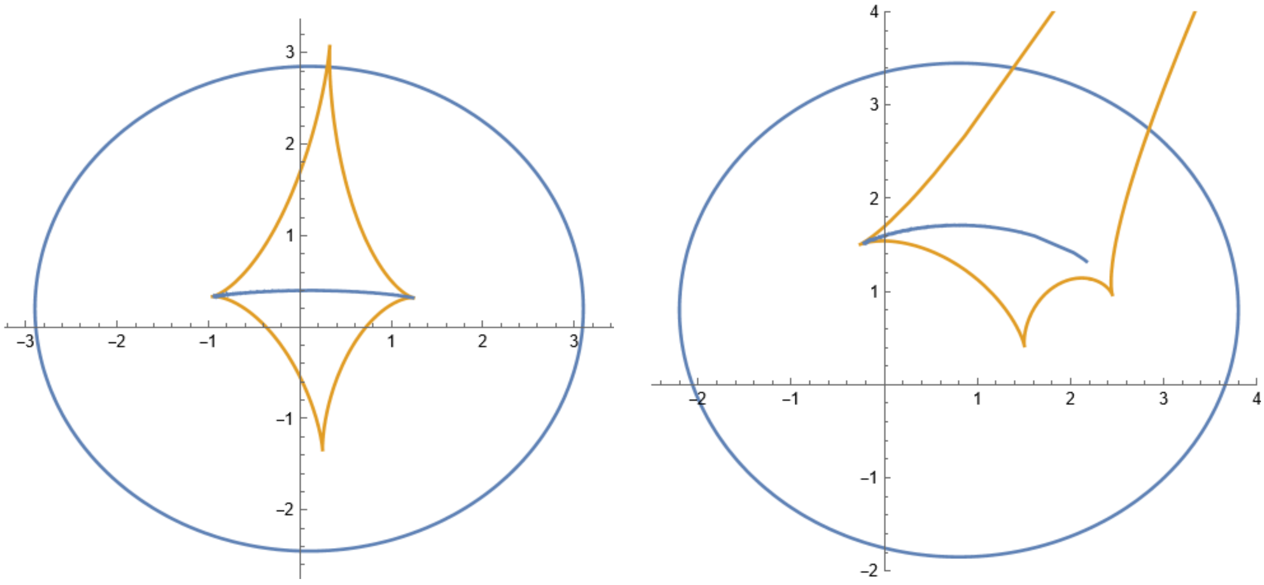
$$\Rightarrow \quad w_2 = \frac{(\frac{y_3 - y_1}{y_2 - y_1}) - (\frac{x_3 - x_1}{x_2 - x_1})}{(\frac{x_4 - x_3}{x_2 - x_1}) - (\frac{y_4 - y_3}{y_2 - y_1})} \quad \Rightarrow \quad \boxed{w_2 = \frac{(y_3 - y_1)(x_2 - x_1) - (x_3 - x_1)(y_2 - y_1)}{(x_4 - x_3)(y_2 - y_1) - (y_4 - y_3)(x_2 - x_1)}}$$

$$\Rightarrow \quad w_1 = (\frac{x_3 - x_1}{x_2 - x_1}) + (\frac{x_4 - x_3}{x_2 - x_1})\frac{(y_3 - y_1)(x_2 - x_1) - (x_3 - x_1)(y_2 - y_1)}{(x_4 - x_3)(y_2 - y_1) - (y_4 - y_3)(x_2 - x_1)}$$

$$\Rightarrow \quad \boxed{w_1 = \frac{(y_3 - y_1)(x_4 - x_3) - (x_3 - x_1)(y_4 - y_3)}{(x_4 - x_3)(y_2 - y_1) - (y_4 - y_3)(x_2 - x_1)}}$$

The process of obtaining w_1 and w_2 has been simplified in the paper so as to show only the most important steps. If the found values satisfy $w_1, w_2 \in [0,1]^2$, then the intersection point of the two respective infinite lines will be in both segments, which implies that the segments will intersect with each other (i.e. the wavefront curve has a self-intersection point there).

Once we have defined how to see if two segments self-intersect, let us explain the whole method. Firstly, we divide a wavefront curve in several segments, then, we compare each segment of the curve with all the remaining segments except with itself, the former one and the next one. Furthermore, once two segments have been compared, they will not be compared again (as this would be time-consuming). Finally, this is done not only for a particular wavefront, but also for all of them.



(a) $\alpha_3(\theta)$ (b) $\alpha_4(\theta)$

Figure 7: Two ellipses with its caustic and cut locus. The ellipses have the same a and b defined in (1), though they are displaced with each other. Recall that the radiant point is at the origin.

Using Mathematica software, we can obtain the self-intersection points, and they can be plotted in the same way as in the previous method. Looking at Figure 7, we can see two plots of the curve with its caustic and cut locus; on the left-hand side image, the plot is very similar to the one for the former method though the cut locus seems to be a bit longer in this case, whereas on the right-hand side, the cut locus seems to be incomplete (this will be explained later). The main advantage of this method is that it takes much less time to find the self-intersection points in comparison with the first method (thus, it is less time-consuming).

The main disadvantage of this method is how we divide the wavefront curve and how this can be an issue at times. There are 2 possible ways: either in equal length segments (for which an arc length parameterization would be required, and this might be impossible to find) or in segments which would be defined for equal length ranges of θ. The second one has been used, however, its main disadvantage is the fact that not all the segments have the same length, in fact, for a wavefront curve with 4 cusps well-seen, the closer to those points, the smaller the segments (see Figure 6).

The reason why this is a disadvantage is because the code sometimes starts classifying those 4 cusps as self-intersection points (which is something inappropriate and disturbing). Indeed, during the process, it was appreciated that this happens with very close and small segments, whereas it was noticeable that the real self-intersection points occurred for segments with a lot of segments in between.

Thus, the approach was changed, and instead of comparing two segments that have at least a segment in between, it was decided to compare two segments with at least six segments in between (this was based on having divided each wavefront in 80 or 100 or 120 segments).

2.3 Common inconveniences in the two methods

The process of obtaining the self-intersection points is time-consuming in both cases and, when the center of the ellipse is farther away from the radiant point, both methods seem not to work as well, since the cut locus does not appear to be complete.
For the second method, as was previously said, the approach of comparing two segments that have at least 6 segments in between was used. The reason was to exclude the fake self-intersection points detected at the

cusps of the wavefronts, however, by doing this we were taking the risk of excluding real self-intersection points as well. If, instead, we compare two segments with at least 5 segments in between, we would appreciate that the cut locus seems to be longer indeed, though it also includes the cusps for some wavefronts.

3 Conclusions

3.1 The confocal conics

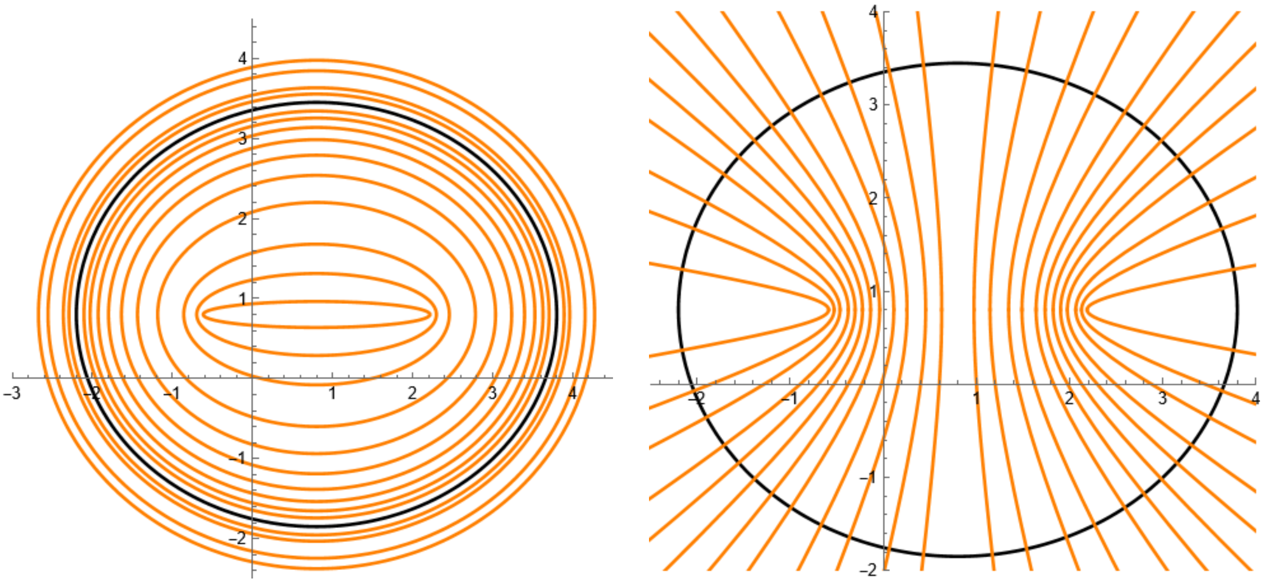
Finally, let us explain the confocal conics to see how they relate to the former sections. If several conic sections have the same foci, then those conics are called confocal [5]. The (confocal) conics that we are interested in in this paper are the ellipse and the hyperbola.

The foci F_1, F_2 determine two families of confocal ellipses and hyperbolae. A common representation specifies a family of ellipses and hyperbolae confocal with a given ellipse of semi-major axis a and semi-minor axis b (so that $0 < b < a$), each conic generated by choice of the parameter λ [5]. We will be using this last point as we want to analyze the caustics of ellipses.

We define the family of confocal conics to an ellipse, and for the general ellipse previously defined in (1), it is given by [1,5,6] (adapted to an ellipse whose centre is not in the origin):

$$\frac{(x-x_0)^2}{a^2-\lambda}+\frac{(y-y_0)^2}{b^2-\lambda}=1 \tag{7}$$

Looking at (7), it will be a confocal ellipse if $\lambda < b^2$ and a confocal hyperbola if $b^2 < \lambda < a^2$, whereas for $\lambda > a^2$ there are no solutions (see Figure 8). For a confocal ellipse, if $0 < \lambda < b^2$, then the confocal ellipse is inside the given ellipse, if $\lambda = 0$, then the confocal ellipse completely overlaps the given ellipse, and if $\lambda < 0$, then the confocal ellipse is outside the given ellipse (this explanation appears in [1,5,6]). The common foci of every conic in the family are the points $\left(x_0 \pm \sqrt{a^2-b^2},\ y_0\right)$ for $a \geq b$ [7].



(a) Family of ellipses **(b)** Family of hyperbolae

Figure 8: Families of confocal conics for the same ellipse $\alpha_4(\theta)$

As the parameter λ approaches the value b^2 from below, the limit of the family of confocal ellipses degenerates to the line segment between foci on the horizontal axis $y = y_0$ (an infinitely flat ellipse). As λ approaches b^2 from above, the limit of the family of confocal hyperbolae degenerates to the relative complement of that line segment with respect to the horizontal axis $y = y_0$; that is, to the two rays with endpoints at the foci pointed outward along the horizontal axis $y = y_0$ (an infinitely flat hyperbola). These two limit curves have the two foci in common [5].

Looking at Figure 8, we can appreciate that for values of λ closer to b^2 from below, the confocal ellipses start to squash to the $y = y_0$ axis (8a), whereas for values closer to b^2 from above, the confocal hyperbolae start to squash to the $y = y_0$ axis as well (8b), as explained earlier.

3.2 The final results

After explaining the confocal conics and having already found the self-intersection points, we are now in the situation to show all the results altogether in a more complete plot (having elements from the previous sections and from this section). The intention is to see how related the radiant point, the caustic, the cut locus and the confocal conics are.

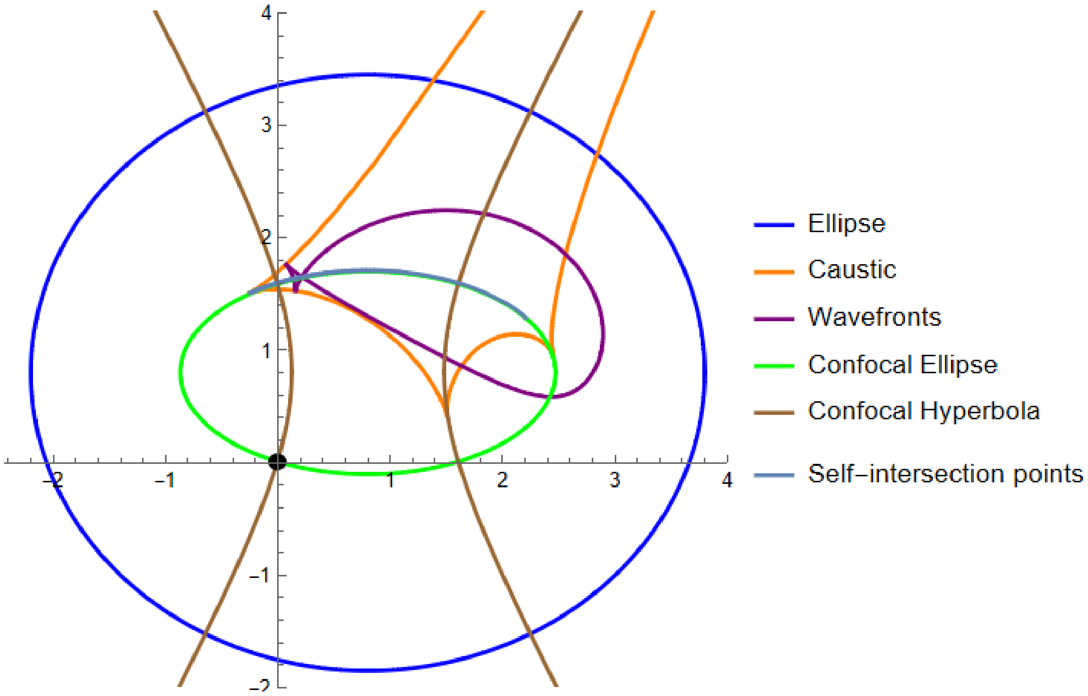


Figure 9: For ellipse $\alpha_4(\theta)$:the confocal hyperbola matches with 2 cusps and the confocal ellipse matches with the other 2 cusps.

Looking at Figure 9, there are many relationships among the aforementioned concepts. First of all, we can see that the radiant point (at the origin) defines a particular pair of a confocal ellipse and a confocal hyperbola, which altogether match at the radiant point. In fact, at the four different points where that pair of confocal conics match each other, it agrees with the fact that the family of confocal ellipses and the family of confocal hyperbolae form a double foliation so that through each point pass one confocal ellipse and one confocal hyperbola, intersecting orthogonally at the point [1,6].

Another characteristic to point out is that that specific confocal hyperbola also matches 2 of the cusps of the caustic, whereas that particular confocal ellipse matches the other 2 cusps of the caustic. Indeed, Theorem 1 from [1] indicates that these 4 cusps of the caustic are the 4 tangency points with the confocal conics through the radiant point (which satisfies being a non-confocal point inside the original ellipse) of the 4 reflected rays emanating from the radiant point tangent to these conics.

Finally, the main contribution of this paper is the fact that part of this specific confocal conic overlaps with the cut locus of the caustic, which is simultaneously formed by the self-intersection points of the first set of wavefronts. Therefore, the results of this analysis would support the conjecture that the cut locus is a curve segment of the specific confocal ellipse that matches the radiant.

3.3 Further questions

Throughout this paper, we have been looking at ellipses, the self-intersections points of their wavefronts and how all of that is related with the cut locus and the caustic. However, we should ask some questions if we want to study problems of this type more deeply.

Ellipses have some nice properties: they are closed, simple and convex curves, which make them easier to study than other curves. However, what about the curves that appear in Figure 2, such as a curve that is closed, simple, and not convex but at least star-shaped with respect to the radiant point (2a)? And what about a limaçon / Pascal's Snail (2b), which is closed but not simple? (**Note:** Figure (2a) has 7 cusps, not 4).

During Section 2, we have applied 2 methods for finding self-intersection points of any plane curve, but it would be of interest to apply other methods because: would they have been faster or slower? Would they have produced the whole cut locus for an ellipse whose center is farther away from the radiant (look at Figure 7, which was obtained with the second method)? Would they have been useful to consider curves that are not ellipses?

I hope this paper can help us to better understand and make a significant contribution to the world of caustics.

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